Chapter 9 - Combinatorics

9.1 Combinatorics

Combinatorics is the branch of mathematic that is primarily concerned with counting.

Fundamental Counting Principle (The Multiplication Principle)

- If we can perform a first task in a different ways
- If we can perform a second task in b different ways
- If we can perform a third task in c different ways, and so on…

Then the first task followed by the second and so on can be performed in

$\boldsymbol{a \cdot b \cdot c}$… different ways

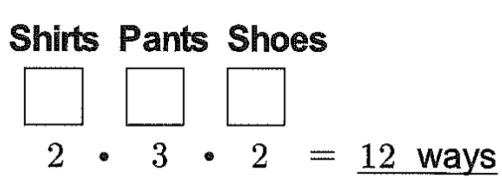
These tasks do not affect each other; they are called **independent events**.

Examples Using the Multiplication Principle

Ex. A man has 2 shirts (red and yellow), 3 pairs of pants (black, white, and navy), and 2 pairs of shoes (green and orange). How many different outfits can he wear?

There are 2 choices of shirts. For each choice of shirts, there are 3 choices of pants. So, there are 2×3 possible combinations for shirts and pants. For each combination of shirts and pants, there are 2 choices of shoes.

Therefore, the possible number of outfits is $2 \times 3 \times 2 = 12$



Ex. A particular automobile has 4 different models, 3 sizes of motors and 6 colour schemes. How many different ways can an automobile be ordered?

Models Motors Colours

$$4 \times 3 \times 6 = 72$$

There are 72 different ways

Ex. The first 4 questions on a quiz are true-false questions, while the next 6 questions are multiple choice with possible answers a, b, c, d, and e. How many different possible answer sequences are there for these 10 questions?

T/F: 2 x 2 x 2 x 2 = 2^4 = 16

M/C: 5 x 5 x 5 x 5 x 5 x 5 = 5^6 = 15625

Total = T/F × M/C

= 16 × 15625

= 250000

250,000 possible sequences

Ex. How many telephone numbers are available with the 778 prefix?

After the 778 prefix, each of the 7 numbers have 10 possibilities, assuming no restrictions.

7	7	8	-	___	___	___	___	___	___	___
				#1	#2	#3	#4	#5	#6	#7

$$10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10$$

$10^7 = 10{,}000{,}000$ possible phone numbers

Ex. How many different ways can 5 different books be arranged on a shelf?

The first book would have 5 choices. With 4 books left, the second choice would have 4 choices. Then the third book with have 3 choices, while the fourth and fifth books would have 2 and 1 choices respectively.

$5 \times 4 \times 3 \times 2 \times 1 = 120$ different ways

Ex. How many different can 3 books be arranged on a shelf, using 5 different books?

$5 \times 4 \times 3 = 60$ different ways

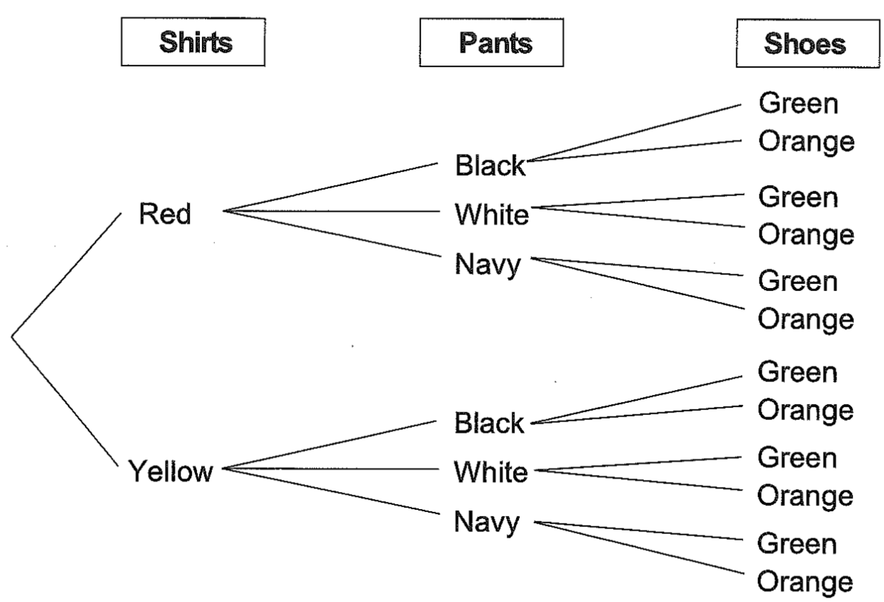
Tree Diagrams

We use tree diagrams to have a systematic way of counting the outcomes.

Examples Using Tree Diagrams

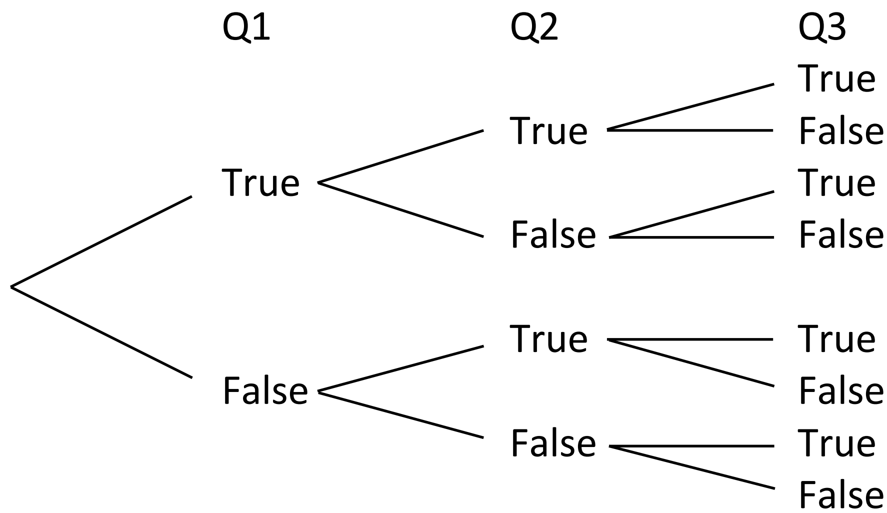
Ex. A man has 2 shirts (red and yellow), 3 pairs of pants (black, white, and navy), and 2 pairs of shoes (green and orange). How many different outfits can he wear?

List out possible outcomes using a tree diagram



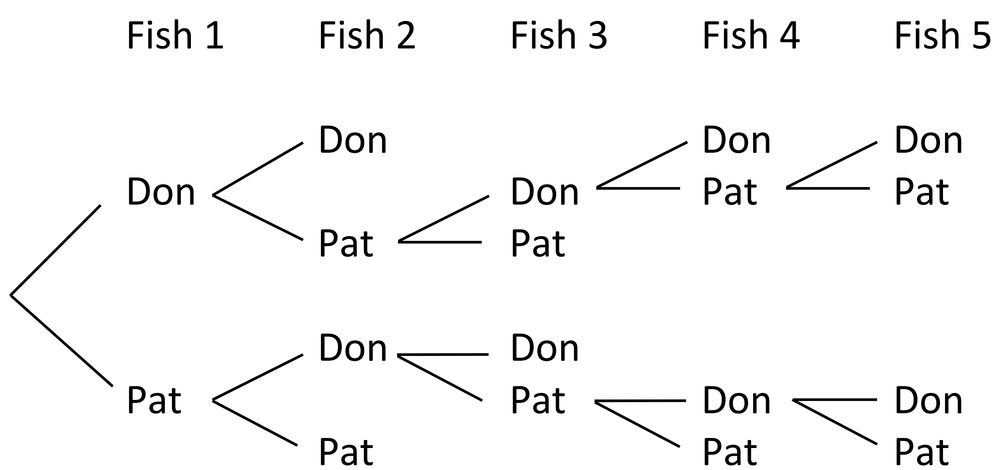
$\therefore$ 12 ways

Ex. How many ways can a 3 item true-false test be answered?



∴ 8 ways

Ex. Don and Pat are in a fishing tournament. The first person to catch 2 fish in a row or 3 fish in total wins the tournament. How many different outcomes are possible?



∴ 10 ways

Outcomes:

DD, PP, DPP, PDD, DPDD, PDPP, DPDPD, DPDPP, PDPDD, PDPDP

Factorial Notation!

The product of the consecutive positive integers from 1 to n is given a special name, n factorial, which is written $n!$ $(n \geq 0)$

$0! = 1$ by definition

$1! = 1$

$2! = 2 \times 1 = 2$

$3! = 3 \times 2 \times 1 = 6$

$4! = 4 \times 3 \times 2 \times 1 = 24$

$\vdots$

$$n! = n(n-1)(n-2) \ldots 3 \cdot 2 \cdot 1$$

Ex. Evaluate 5!

$5!$

$= 5 \times 4 \times 3 \times 2 \times 1$

$= 120$

Ex. How many different "words" can be made using all the letters in abcde?

This is the exact same situation as the problem with arranging 5 books. First book has 5 choices, second book as 4 choices, and etc.

$5! = 120$

There are 120 "words" that can be constructed from abcde

Ex. A three-digit password is made from using numbers 0-9.

a. How many possible passwords are there if there are no restrictions

$$10 \cdot 10 \cdot 10 \cdot 10$$

$$= 10000$$

b. How many possible passwords are there if none of the numbers be repeated?

$$10 \cdot 9 \cdot 8 \cdot 7$$

$$= 5040$$

c. How many possible passwords are there if a number can be used up to 3 times?

$$10 \cdot 10 \cdot 10 \cdot 10 - 10$$

$$= 9990$$

Ex. Simplify and the evaluate $\frac{86!}{84!}$

$\frac{86!}{84!}$ Re-write $86!$ as $86 \times 85 \times 84!$

$= \frac{86 \times 85 \times 84!}{84!}$ Cancel out $84!$ from the numerator and denominator

$= 86 \times 85$

$= 7310$

Ex. Simplify and then evaluate $\frac{20!-18!}{18!}$

$\frac{20!-18!}{18!}$ Factor out 18!

$= \frac{18!(20\times19-1)}{18!}$ Reduce 18! from numerator and denominator

$= 20 \times 19 - 1$

$= 380 - 1$

$= 379$

Ex. Simplify $\frac{(n+1)!}{(n-1)!}$

$\frac{(n+1)!}{(n-1)!}$

$= \frac{(n+1)(n)(n-1)!}{(n-1)!}$

Need to re-write the larger factorial so it has a common factor with the smaller factorial $(n+1)! = (n+1)(n)(n-1)!$

$= (n+1)(n)$

$= n^2 + n$

Ex. Simplify $\frac{n!}{(n-2)!+(n-3)!}$

$= \frac{n(n-1)(n-2)(n-3)!}{(n-2)(n-3)!+(n-3)!}$

$= \frac{n(n-1)(n-2)}{(n-2)+1} = \frac{n(n-1)(n-2)}{n-1}$

$= n(n-2) = n^2 - 2n$

Solving Factorial Equations

Ex. Solve $\frac{n!}{(n-2)!} = 3!$

$\frac{n(n-1)(n-2)!}{(n-2)!} = 6$

$n(n-1) = 6$

$n^2 - n - 6 = 0$

$(n+2)(n-3) = 0$

$n = -2, 3$ reject -2

$n = 3$

Ex. Solve $3!\,(n+1)! = 5!\,(n-1)!$

$6(n+1)(n)(n-1)! = 120(n-1)!$

$(n+1)(n) = 20$ able to cancel out $(n-1)!$, because it $= 1$

$n^2 + n - 20 = 0$

$(n+5)(n-4) = 0$

$n = -5, 4$ reject -5

$n = 4$

9.1 Homework
7.1 pg 451 # 2, 3, 5, 8, 9, 12, 16, 17bd, 18bcf, 19