Ch 3 Polynomials

3.1 Polynomials

A polynomial is an expression: $ax^n + bx^{n-1} + cx^{n-2} + \cdots + D$

Where $a, b, c, \ldots, D$ are real numbers and n is a whole number

a – leading coefficient, non-zero

n – degree of the polynomial

D – constant term

Examples of Polynomial Functions and Characteristics

The degree, leading coefficient, and type of graph is shown.

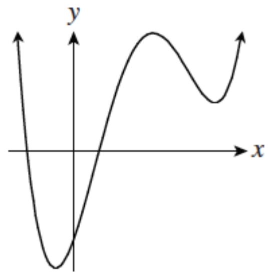
Equation	Degree	LC	Graph / Shape
$f(x) = 4$	0	4	Horizontal Line
$g(x) = \frac{3}{2}x + 5$	1	$\frac{3}{2}$	Linear Function
$y = 9x^2 - 4x$	2	9	Quadratic Function
$d = -7x^3 + x - 1$	3	-7	Cubic Function
$y = \sqrt{3}x^4 - 5x^2 + \sqrt{2}$	4	$\sqrt{3}$	Quartic Function

Examples of Non-Polynomials and Reason

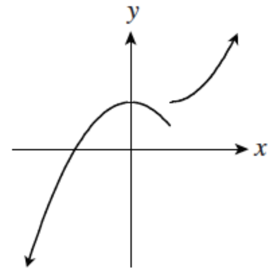
Equation	Reason
$f(x) = \sqrt{3x^3} + x$	First term degree is $\frac{3}{2}$; not a whole number
$y = \sqrt{-3}t^2 + 5t - 4$	$\sqrt{-3}$ is not a real number
$f(x) = 5x^{-3} + 1$	One of the variables has a negative exponent
$g(x) = \frac{3}{x^2} + 4x - x^2$	One of the variables has a negative exponent

Graphs of Polynomials

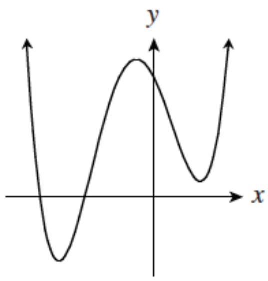
All polynomial functions are **smooth** curves. Polynomial graphs do not have corners, gaps, nor cusps.



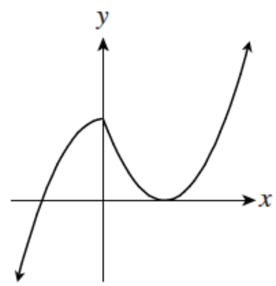
Polynomial function



Non-polynomial function – has a break



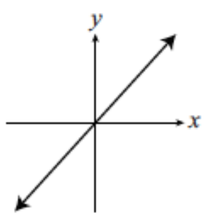
Polynomial function



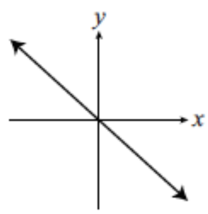
Non-polynomial function – has a corner

Shape of Common Polynomial Functions, $f(x) = x^n$ and $f(x) = -x^n$

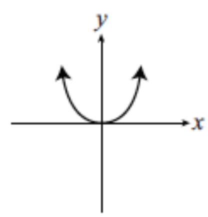
$f(x)=x$



$f(x)=-x$



$f(x)=x^2$



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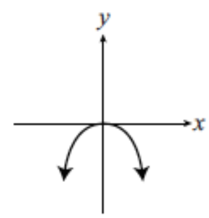
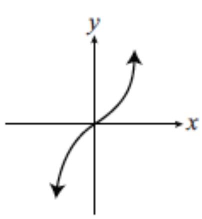
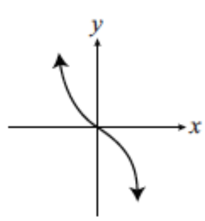


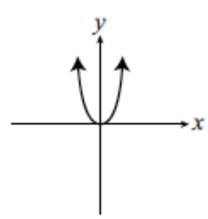
$f(x)=x^3$



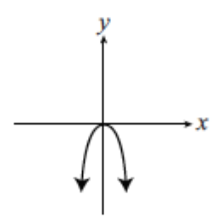
$f(x)=-x^3$



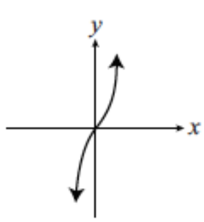
$f(x)=x^4$



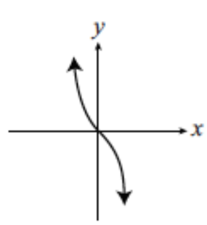
$f(x)=-x^4$



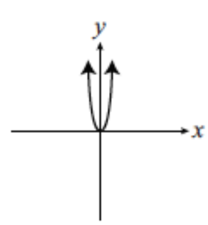
$f(x)=x^5$



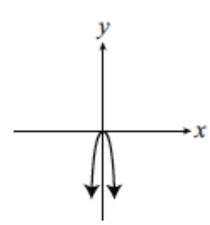
$f(x)=-x^5$



$f(x)=x^6$



$f(x)=-x^6$



End Behaviour

The end behaviour of polynomial functions can be predicted by analyzing its degree and leading coefficient.

When the degree is **odd**, the left and right end go in **opposite** directions

If the **leading coefficient** is **positive**, $\boldsymbol{f(x) = x^n}$**:**

$x \rightarrow left$ $\quad y \rightarrow down$

$x \rightarrow right$ $\quad y \rightarrow up$

If the **leading coefficient** is **negative,** $\boldsymbol{f(x) = -x^n}$**:**

$x \rightarrow left$ $\quad y \rightarrow up$

$x \rightarrow right$ $\quad y \rightarrow down$

When the degree is **even**, the left and right end go the **same** direction

If the **leading coefficient** is **positive**, $\boldsymbol{f(x) = x^n}$**:**

$x \rightarrow left$ $\quad y \rightarrow up$

$x \rightarrow right$ $\quad y \rightarrow up$

If the **leading coefficient** is **negative,** $\boldsymbol{f(x) = -x^n}$**:**

$x \rightarrow left$ $\quad y \rightarrow down$

$x \rightarrow right$ $\quad y \rightarrow down$

Ex. Determine the end behaviour of following functions.

a. $y = -3x^4 + 2x^2 - 1$

Even degree and negative LC

$x \rightarrow left$ $\quad y \rightarrow down$

$x \rightarrow right$ $\quad y \rightarrow down$

b. $f(x) = 8 - x^5$

Odd degree and negative LC

$x \rightarrow left$ $\quad y \rightarrow up$

$x \rightarrow right$ $\quad y \rightarrow down$

Constant Value of a Polynomial Function, y −intercept

The constant term in a polynomial function is the y-intercept.

$$f(x) = ax^n + bx^{n-1} + \cdots + D$$

The **constant value D** of a polynomial is the **y-intercept**.

Ex. For the following polynomial functions, determine the y-intercept.

a. $f(x) = 3x^4 + 5x^2 - x + 11$

The y-int is 11

b. $f(x) = -11x^5 + 4x^3 - 2x$

The y-int is 0

c. $f(x) = ax^3 + bx^2 + cx + d$

The y-int is d

Zeros of a Polynomial Function, x −intercepts

Zeros, roots, x −int, and solutions all refer to the same thing.

Maximum Number of Possible Solutions

For odd and even degree functions, the **maximum** number of zeros, is equal to its degree.

Minimum Number of Possible Solutions

For **odd** degree functions, the **minimum** number of solutions is **1**.
For **even** degree functions, the **minimum** is **no solutions**.

Ex. Determine the max and min number of solutions for the following.

a. $y = -12x^7 + 8x - 1$

Max = 7, Min = 1

b. $f(x) = 25x^2 + 50x - 2$

Max = 2, Min = 0

Zeros and Multiplicity of the root

Multiplicity of a root is the **number of times a root occurs.**

For $f(x) = (x - a)(x - b)^2(x - c)^3$

The solution at $x = a$ has a multiplicity of 1
$x = b$ has a multiplicity of 2
$x = c$ has a multiplicity of 3

Ex. Solve $3x(x - 2)(x - 5) = 0$ and state the multiplicity of each root.

$x = 0, 2, 5$

Since each answer occurred once, they have a *multiplicity* of 1.

Ex. Solve $(2x - 3)(x + 2)^2(x - 1) = 0$ and state the multiplicity of each root.

$x = \frac{3}{2}, -2, 1$

Solution at $x = \frac{3}{2}$ and $x = 1$ have a multiplicity of 1, while $x = -2$ has a multiplicity of 2.

Ex. Solve $x^3 - 4x^2 - x + 4 = 0$ and state the multiplicity of each root.

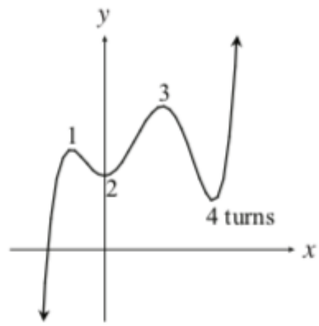
First, factor the polynomial and then solve.

$x^2(x - 4) - 1(x - 4) = 0$
$(x - 4)(x^2 - 1) = 0$
$(x - 4)(x + 1)(x - 1) = 0$

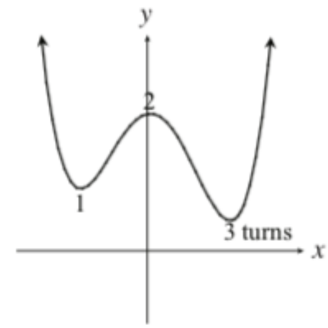
$x = -1, 1, 4$ Each has multiplicity of 1

Turning Points

The **minimum degree** of a polynomial is equal to **the number of turning points plus 1**.



Minimum degree of this polynomial function is 5.

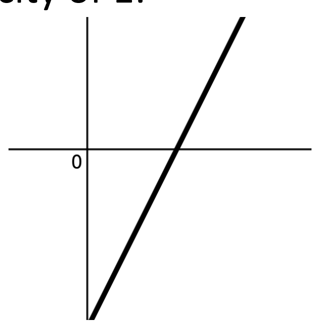


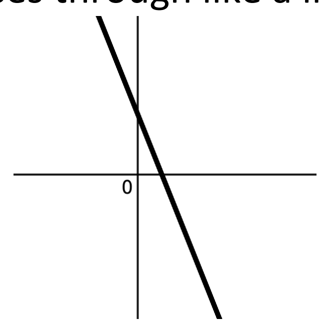
Minimum degree of this polynomial function is 4.

Multiplicity and shape of the graph

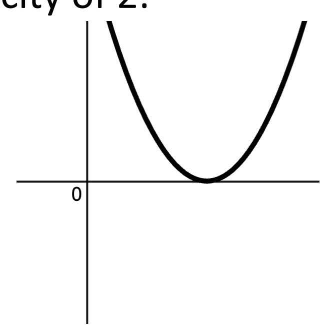
The multiplicity of a solution determines the behaviour of the graph at the x-intercept.

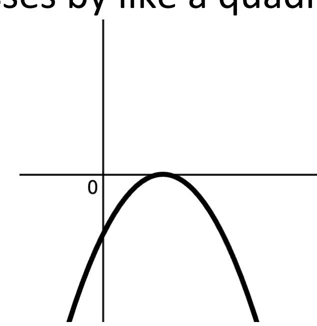
Multiplicity of 1: passes through like a linear graph



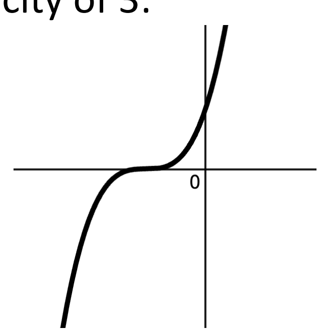


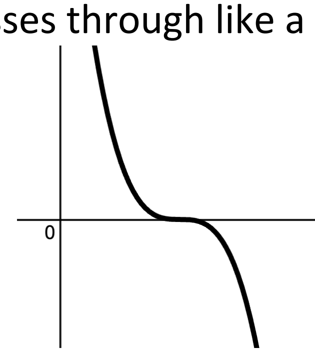
Multiplicity of 2: passes by like a quadratic (parabola)





Multiplicity of 3: passes through like a cubic graph





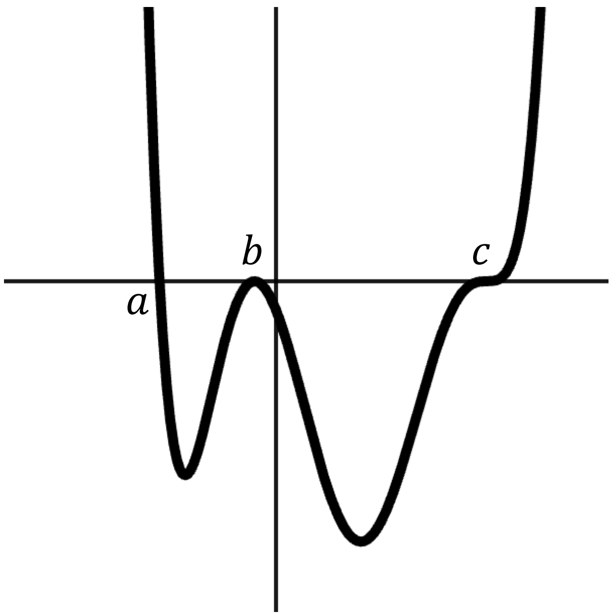
Sketch a Polynomial Using the Degree, Leading Coefficient, and Multiplicity

Ex. Sketch $f(x) = (x-a)(x-b)^2(x-c)^3$, where $a < b < 0 < c$.

The degree of the polynomial is $1 + 2 + 3 = 6$

If $f(x)$ were to be simplified, the leading term would be x^6

Even degree and positive LC means the left and right end point up



$x = a$	has multiplicity of 1 passes through linearly
$x = b$	has multiplicity of 2 touches the x-axis but does not pass through
$x = c$	has multiplicity of 3 passes through like a cubic function

3.1 Homework:
2-5, 6-7 bcf…, 8bc, 9 bcf…

3.2 Graphing Polynomial Functions

Sketching Polynomial Functions With Approximate Turning Points

The steps to sketching a polynomial function:

1. Determine all the intercepts:
   Solving $f(x) = 0$, determine the x-intercepts
   Finding the y-intercept by evaluating $y = f(0)$

2. Determine the shape of each x-intercept; determine the multiplicity of each solution

3. Using the leading coefficient and the degree, determine the end behaviour

4. Estimate the turning points (local max / min). These will occur at double (even) roots or in between two x intercepts

5. Plot all coordinates, and draw a smooth curve connecting all coordinates

Ex. Sketch $f(x) = x(x+1)^2(x-2)(x-4)$

1. Find x and y intercepts

   $f(x) = 0$
   $x(x+1)^2(x-2)(x-4) = 0$
   $x = 0, -1, 2, 4$

   $f(0) = 0$
   The y-int is at $(0, 0)$

2. Determine the multiplicity of each root

   $x = 0, M = 1$ $\qquad$ $x = -1, M = 2$

   $x = 2, M = 1$ $\qquad$ $x = 4, M = 1$

3. Leading term will be x^5

LC > 0 — no vertical reflection (Leading Coefficient)

Deg = 5 — odd degree: opposite directions

End Behavior: $x \to left, y \to down$

$x \to right, y \to up$

4. Estimate Turning points (TP)

TP at the double root at -1

TP_1: $(-1, 0)$

TP in between the roots at -1 and 0

$f(-0.5) = -1.4$

TP_2: $(-0.5, -1.4)$

TP in between the roots at 0 and 2

$f(1) = 12$

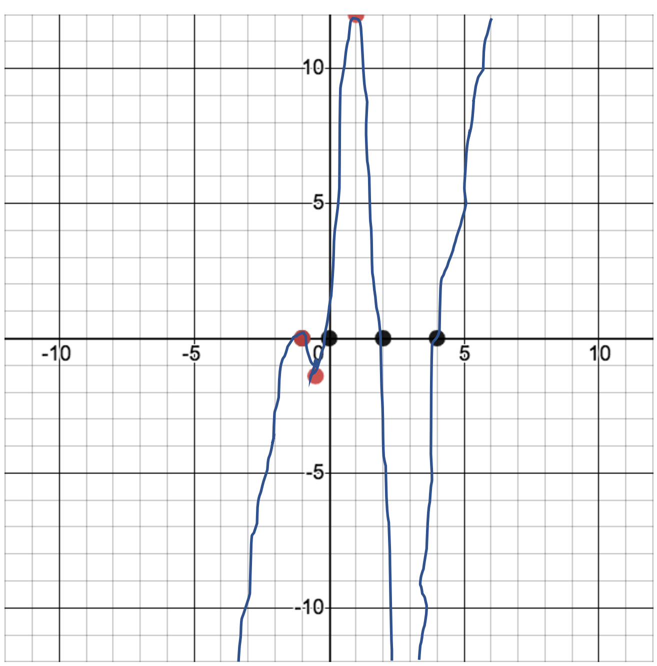
TP_3: $(1, 12)$

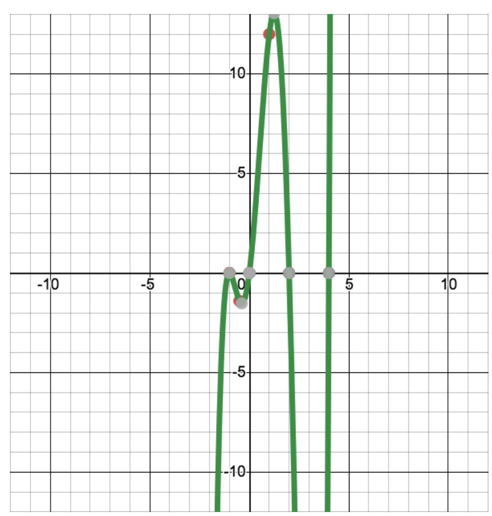
TP in between the roots at 2 and 4

$f(3) = -48$

TP_4: $(3, -48)$

5. Plot all coordinates and draw a smooth curve





(Actual Graph)

Ex. Sketch $f(x) = x^4 - 3x^3 - 4x^2$.

1. $x^2(x^2 - 3x - 4) = 0$
   $x^2(x - 4)(x + 1) = 0$
   $x = 0, 4, -1$
   $f(0) = 0$ (y-int)

2. $x = 0, M = 2$
   $x = 4, M = 1$
   $x = -1, M = 1$

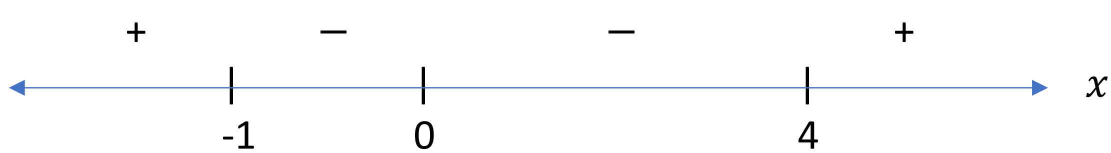
3. LC > 0, no vertical reflection
   Degree = 4 even degree: same direction
   End Behavior:
   $x \to left, y \to up$
   $x \to right, y \to up$

4. TP between -1 and 0
   $f(-0.5) = -0.6$
   TP_1: $(-0.5, -0.6)$

   Double root at $x = 0$
   TP_2: $(0, 0)$

   TP between 0 and 4
   $f(2) = -24$
   TP_3: $(2, -24)$

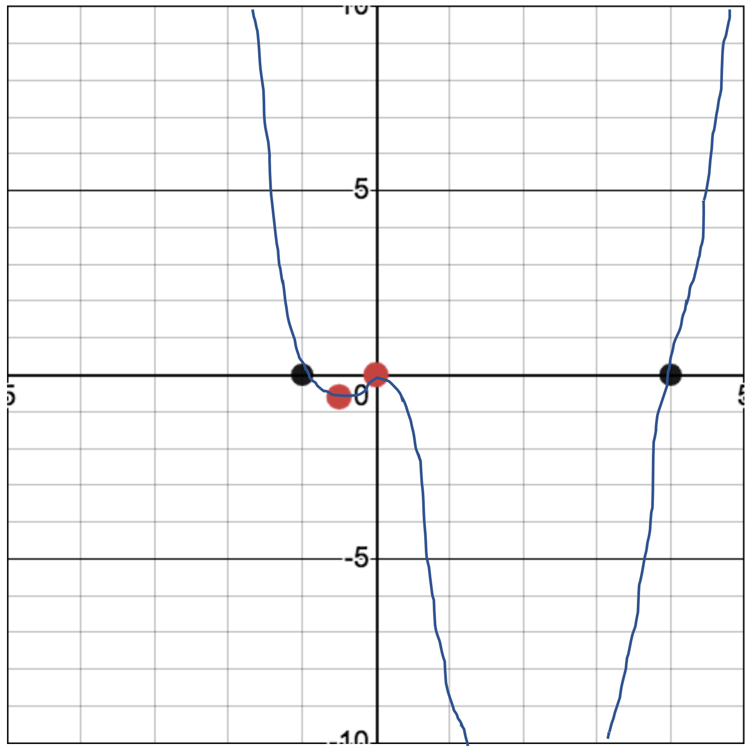
   Use interval testing, we can determine where the graph will be above or below the x-axis

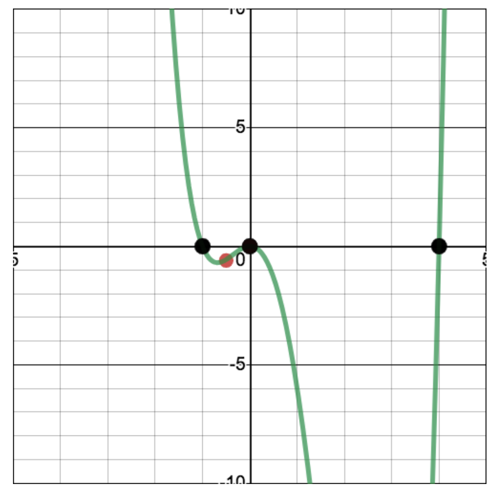


$f(-2) = 24$ $f(-0.5) = -0.6$

$f(2) = -24$ $f(5) = 150$

5. Sketch the curve





Determine the Equation of a Polynomial

Ex. A polynomial has -1, -1, 0, 2 as its roots, and $f(1) = 5$. Determine the equation of the polynomial.

Write the equation with the given roots in factored form. Include a in the front to adjust for vertical expansion/compression.

$$f(x) = a(x)(x+1)^2(x-2)$$

Now, solve for a.

Using $f(1) = 5$:

$$5 = a(1)(1+1)^2(1-2)$$
$$5 = -4a$$
$$a = -\frac{5}{4}$$

$$\therefore f(x) = -\frac{5}{4}x(x+1)^2(x-2)$$

3.2 Homework:

1-3, 4 bcf…, 5, 6, 7 bcf, 8

3.3 Polynomial Division

Polynomial Division

When a polynomial $P(x)$ is divided by a divisor $d(x)$, the result can be written in one of two forms.

$$\frac{P(x)}{d(x)} = Q(x) + \frac{r(x)}{d(x)} \quad \text{or} \quad P(x) = Q(x)d(x) + r(x)$$

For the above polynomial division:

$P(x)$	a polynomial, “the dividend”
$d(x)$	the divisor
$Q(x)$	the quotient, “the answer”
$r(x)$	the remainder, “the remainder”

This similar to division statements learned in previous grades. For example, 7 divided by 2 is equal to 3 with a remainder of 1. The result can be written as:

$$\frac{7}{2} = 3 + \frac{1}{2} \quad \text{or} \quad 7 = (3)(2) + 1$$

Types of Polynomial Division

For polynomial division, there are 2 main methods to carry out calculations:

1. Long Division
2. Synthetic Division

A combination of the two methods above yields a “hybrid method”:

3. Synthetic-Long Division

Computing Polynomial Division Using Polynomial Long Division

Ex. Divide $(2x^3 - 7x^2 + 4x - 3) \div (x + 2)$ using polynomial long division.

$$x + 2\,\overline{)\,2x^3 - 7x^2 + 4x - 3}$$

Determine the number of times x "goes into" $2x^3$: $\frac{2x^3}{x} = 2x^2$

Then multiply it to the divisor and subtract it from the dividend

$$\begin{array}{rl} & \quad 2x^2 \\ x + 2 & \overline{)\,2x^3 - 7x^2 + 4x - 3} \\ - & \underline{(\,2x^3 + 4x^2\,)} \\ & \quad -11x^2 \end{array}$$

Drop down the next term

$$\begin{array}{rl} & \quad 2x^2 \\ x + 2 & \overline{)\,2x^3 - 7x^2 + 4x - 3} \\ & -\underline{(\,2x^3 + 4x^2\,)} \quad \downarrow \\ & \quad -11x^2 + 4x \end{array}$$

Then repeat the process above

$$\begin{array}{rl} & \quad 2x^2 - 11x \\ x + 2 & \overline{)\,2x^3 - 7x^2 + 4x - 3} \\ & -\underline{(\,2x^3 + 4x^2\,)} \\ & \quad\quad -11x^2 + 4x \\ & \quad -\underline{(\,-11x^2 - 22x\,)}\downarrow \\ & \quad\quad\quad\quad -26x - 3 \end{array}$$

$$\begin{array}{rl} & \quad 2x^2 - 11x + 26 \\ x + 2 & \overline{)\,2x^3 - 7x^2 + 4x - 3} \\ & -\underline{(\,2x^3 + 4x^2\,)} \\ & \quad -11x^2 + 4x \\ & \quad -\underline{(\,-11x^2 - 22x\,)} \\ & \quad\quad\quad\quad 26x - 3 \\ & \quad\quad\quad -\underline{(\,26x + 52\,)} \\ & \quad\quad\quad\quad\quad -55 \end{array}$$

To finish, write the result in one of the two forms shown at the beginning of the section.

$$\therefore \frac{2x^3-7x^2+4x-3}{(x+2)} = 2x^2 - 11x + 26 - \frac{55}{x+2}$$

Or

$$2x^3 - 7x^2 + 4x - 3 = (2x^2 - 11x + 26)(x + 2) - 55$$

Polynomial Division With "Gaps" or Missing Terms

For polynomials with missing "x" terms, fill it with a $0x$. For missing "x^2", fill it with a $0x^2$ and et cetera.

For example, $3x^3 - 2x^2 + 5$ becomes $3x^3 - 2x^2 + 0x + 5$

Ex. Divide $(3x^3 - 2x^2 + 5) \div (x + 3)$ using polynomial long division.

Set Up:

$$x + 3\,\overline{)\,3x^3 - 2x^2 + 0x + 5}$$

$$\begin{array}{r l}
 & \quad 3x^2 - 11x + 33 \\
x + 3 \,) & \overline{3x^3 - 2x^2 + 0x + 5} \\
 & -\underline{(3x^3 + 9x^2)} \\
 & \quad -11x^2 + 0x \\
 & \quad -\underline{(-11x^2 - 33x)} \\
 & \qquad\qquad 33x + 5 \\
 & \qquad\qquad -\underline{(33x + 99)} \\
 & \qquad\qquad\qquad -94
\end{array}$$

$$\therefore 3x^3 - 2x^2 + 5 = (3x^2 - 11x + 33)(x + 3) - 94$$

Computing Polynomial Division Using Polynomial Synthetic Division

Ex. Divide $(2x^3 - 7x^2 + 4x - 3) \div (x + 2)$ using Synthetic Division.

For the general divisor $(x - a)$,

$$x - a = 0$$
$$x = a$$

$\therefore$ the divisor number is a.

Since the divisor is $(x + 2)$,

$$x + 2 = 0$$
$$a = -2$$

$\therefore$ the divisor number is -2

So $a = -2$

In general, this is the set up for synthetic division

$$\begin{array}{r|l} a & \text{Coeffificents of } P(x) \\ \hline & \end{array}$$

Next, fill in the synthetic division with the coefficients of $P(x)$

$$\begin{array}{r|rrrr} -2 & 2 & -7 & 4 & -3 \\ & & & & \\ \hline & & & & \end{array}$$

To begin the synthetic division process, drop down the first coefficient

$$\begin{array}{r|rrrr} -2 & 2 & -7 & 4 & -3 \\ & \downarrow & & & \\ \hline & 2 & & & \end{array}$$

Next, multiply the first coefficient by the divisor number

$$\begin{array}{r|rrrr} -2 & 2 & -7 & 4 & -3 \\ & & -4 & & \\ \hline & 2 & & & \end{array}$$

Add the next column of numbers together, and put the result in the bottom row after the first coefficient

-2	2	-7	4	-3
		-4		
	2	-11		

Repeat the process until the end

-2	2	-7	4	-3
		-4	22	
	2	-11		

-2	2	-7	4	-3
		-4	22	
	2	-11	26	

-2	2	-7	4	-3
		-4	22	-52
	2	-11	26	

-2	2	-7	4	-3
		-4	22	-52
	2	-11	26	-55

The process is now complete, interpret the solution:

-2	2	-7	4	-3
		-4	22	-52
	2	-11	26	-55

Divisor	Quotient	Remainder
$x+2$	$2x^2-11x+26$	-55

$$\therefore 2x^3-7x^2+4x-3=(2x^2-11x+26)(x+2)-55$$

Ex. Divide $(3x^3 - 2x^2 + 5) \div (x + 3)$ using Synthetic Division.

-3	3	-2	0	5
		-9	33	-99
	3	-11	33	-94

$$3x^3 - 2x^2 + 5 = (3x^2 - 11x + 33)(x + 3) - 94$$

Synthetic Long Division

This form of long division is done with only the coefficients

Ex. Divide $(3x^3 - 2x^2 + 5) \div (x + 3)$ using Synthetic Long Division.

$$\begin{array}{llrrrr}
 & & 3 & -11 & 33 & \\
1 & 3\,) & 3 & -2 & 0 & 5 \\
 & & -(3 & 9) & & \\
 & & & -11 & 0 & \\
 & & & -(-11 & -33) & \\
 & & & & 33 & 5 \\
 & & & & -(33 & 99) \\
 & & & & & -94
\end{array}$$

$$\therefore 3x^3 - 2x^2 + 5 = (3x^2 - 11x + 33)(x + 3) - 94$$

Ex. Divide $(3x^3 + 4) \div (x^2 - 1)$

We cannot use synthetic division because the divisor is not linear; it is not in the form of $x - a$. Also, fill in all missing terms with a “0”, even in the divisor.

$$\begin{array}{llrrrrr}
 & & & & 3 & 0 \\
1 & 0 \; -1\,) & 3 & 0 & 0 & 4 \\
 & & -(3 & 0 & -3) & \downarrow \\
 & & & 0 & 3 & 4 \\
 & & & -(0 & 0 & 0) \\
 & & & & 3 & 4
\end{array}$$

$$\therefore (3x^3 + 4) = (3x)(x^2 - 1) + (3x + 4)$$

Using Polynomial to Solve for Unknown Coefficients

Ex. Solve for k, given that the remainder of the following long division is 5.
$(x^3 + kx^2 - 2x - 7) \div (x + 1)$

First find divisor number a

$x + 1$
$x - (-1)$
$\therefore a = -1$

Next, use Synthetic Division to create an equation

-1	1	k	-2	-7
	↓	-1	$-k$+1	k+1
	1	k-1	$-k$-1	5

From the last column, we get:

$$-7 + k + 1 = 5$$
$$k - 6 = 5$$
$$k = 11$$

Alternatively, we can work backwards starting from the last column:

-1	1	k	-2	-7
	↓	-1	-10	12
	1	10	-12	5

From the second column, we get:

$$k + (-1) = 10$$
$$k - 1 = 10$$
$$k = 11$$

3.3 Homework:
#1-4 bcf…

3.4 The Remainder and Factor Theorem

Deriving The Remainder Theorem

Recall: $f(x) = (x - a)q(x) + r$

In synthetic division, from the divisor $x - a$, the divisor number is a

What happens if we evaluate $f(a)$?

$$f(a) = (a - a)q(x) + r$$

$$f(a) = (0)q(x) + r$$

$$f(a) = r$$

The Remainder Theorem

To find the remainder of a polynomial division, $f(x) \div (x - a)$, we evaluate $f(a)$ where a is from the divisor $x - a$.

For $P(x) \div (x - a)$, the remainder r, can be found by: $r = P(a)$

Find Remainder of Polynomial Division Using the Remainder Theorem

Ex. Determine the remainder of $P(x) = 5x^3 - 4x^2 + 3x - 2$ when it is divided by $x + 2$.

The remainder of this polynomial division is equal to $P(a)$; $a = -2$

$$r = P(-2)$$

$$r = 5(-2)^3 - 4(-2)^2 + 3(-2) - 2$$

$$r = -64$$

$\therefore$ the remainder is equal to -64

Ex. When $kx^3 + mx^2 + x - 2$ is divided by $x - 1$, the remainder is 6. When this polynomial is divided by $x + 2$, the remainder is 12. Solve for k and m.

Divisor: $x - 1 \quad \rightarrow \quad a = 1$

$P(1) = 6 \qquad P(1) = k + m + 1 - 2 = m + k - 1$

$m + k - 1 = 6$
$m + k = 7 \qquad (1)$

Divisor: $x + 2 \quad \rightarrow \quad a = -2$

$P(-2) = 12 \qquad P(-2) = -8k + 4m - 2 - 2$
$P(-2) = 4m - 8k - 4$

$4m - 8k - 4 = 12$
$4m - 8k = 16$
$m - 2k = 4 \qquad (2)$

Solve the system:
$m + k = 7$
$m - 2k = 4$

Isolate m from (1)
$m = 7 - k \qquad (1)$

Sub $m = 7 - k$ into (2), and solve for k
$7 - k - 2k = 4$ (3)
$-3k = -3$
$k = 1$

Sub $k = 1$ into (1) to solve for m
$m = 7 - 1$
$m = 6$

$\therefore m = 6, k = 1$

Remainder Theorem With Remainder of Zero

Ex. Find the remainder of $(x^3 - 2x^2 + x - 2) \div (x - 2)$.

Divisor $x - 2$ → $a = 2$

$P(2) = 8 - 8 + 2 - 2 = 0$

Now, carry out the polynomial division.

```
2   |   1   -2   1   -2
    |        2   0    2
-------------------------------
        1    0   1 |  0
```

$\therefore x^3 - 2x^2 + x - 2 = (x - 2)(x^2 + 1)$

Turns out that if the remainder of a polynomial division is equal to zero, then the divisor is a factor of the polynomial.

The Factor Theorem

For a polynomial $f(x)$, when it is divided by $x - a$ and $f(a) = 0$, we say that $x - a$ is a factor of $f(x)$.

For a polynomial $P(x)$, if $P(a) = 0$ then $x - a$ is a factor of $P(x)$.

Ex. For $P(x) = 3x^4 + 4x^3 - 3x^2 - 3x - 10$, is $x + 2$ a factor of $P(x)$?

To determine if it is a factor, check if the remainder is equal to 0

Divisor: $x + 2$ → $a = -2$

$$r = P(-2)$$
$$= 3(-2)^4 + 4(-2)^3 - 3(-2)^2 - 3(-2) - 10$$
$$= 0$$

The **Factor Theorem** states if the remainder of a polynomial division is 0, then the divisor is a factor.

$\therefore x + 2$ is a factor

Using the Factor Theorem to Factor a Polynomial

Ex. Factor the polynomial $P(x) = 2x^4 - 7x^3 + 9x^2 - 5x + 1$.

Guess possible factors: Look at constant term

Try $x = 1$ → factor: $(x - 1)$

$P(1) = 0$

Use synthetic or long division to find other factors

```
1   |   2    -7    9    -5    1
    |         2   -5     4   -1
-----------------------------------
        2    -5    4    -1  |  0
```

$\therefore P(x) = (x - 1)(2x^3 - 5x^2 + 4x - 1)$

Try $a = 1$ again,

$P(1) = 0$

```
1   |   2    -5    4    -1
    |         2   -3     1
----------------------------
        2    -3    1  |  0
```

$\therefore P(x) = (x - 1)(x - 1)(2x^2 - 3x + 1)$

For the remainder factor $2x^2 - 3x + 1$, we can use more conventional factoring methods like decomposition, or cross method.

$2x^2 - 3x + 1$

$= (2x - 1)(x - 1)$

$P(x) = (x - 1)^2(2x - 1)(x - 1)$

$\therefore P(x) = (x - 1)^3(2x - 1)$

Rational Root Theorem

Determine what are reasonable guesses for the divisor number.

$P(x) = ax^n + \cdots + b$

Possible divisor number = $\frac{\pm factors\ of\ b}{\pm factors\ of\ a}$ where a, b $\in \mathbb{Z}$

Ex. Solve $4x^3 + 12x^2 + 5x - 6 = 0$ by first putting it in factored form.

Possible guesses: $\pm 1, \pm 2, \pm 3, \pm 6, \pm \frac{1}{2}, \pm \frac{1}{4}, \pm \frac{3}{2} \pm \frac{3}{4}$

Let's try: $a = 2$ → Divisor: $x - 2$

$f(2) = 4(2)^3 + 12(2)^2 + 5(2) - 6$

$f(2) = 84 \neq 0$ $\therefore x - 2$ is not a factor

Let's try: $a = -2$ → Divisor: $x + 2$

$f(-2) = 4(-2)^3 + 12(-2)^2 + 5(-2) - 6$

$f(-2) = 0$ $\therefore x + 2$ is a factor

Use Synthetic Division to find other factors

```
-2 | 4     12     5     -6
   |       -8    -8      6
   ------------------------------
     4      4    -3  |  0
```

$\therefore 0 = (x + 2)(4x^2 + 4x - 3)$

Factor $4x^2 + 4x - 3$ with trinomial factoring methods

$0 = (x + 2)(2x - 1)(2x + 3)$

x-int = $-2, -\frac{3}{2}, \frac{1}{2}$

Ex. Factor $x^3 - 8$.

Let's try 2 as a possible divisor number
$f(2) = 8 - 8 = 0$ $\therefore x - 2$ is a factor

Next, use synthetic division to find other factors

```
2 | 1    0    0    -8
  |      2    4     8
-----------------------------
    1    2    4  |  0
```

$= (x-2)(x^2 + 2x + 4)$

Special Factors: Sum and Difference of Cubes

Ex. Factor $a^3 + b^3$ (and $a^3 - b^3$)

$$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$$

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$

Ex. Factor $8(x+2)^3 - (y-1)^3$. Use difference of cubes.

Let $A = x + 2$ and $B = y - 1$

$= 8A^3 - B^3$

$= (2A)^3 - (B)^3$

$= (2A - B)(4A^2 + 2AB + B^2)$

$= (2(x+2) - (y-1))(4(x+2)^2 + 2(x+2)(y-1) + (y-1)^2)$

$= (2x - y + 5)(4x^2 + 2xy + 14x + y^2 + 2y + 13)$

3.4 Homework:

1-5 bcf…

3.5 Polynomial Applications

Ex. Three consecutive odd integers have a product of -105. What are the three integers?

Let $x =$ first odd integer
$x + 2 =$ second odd integer
$x + 4 =$ third odd integer

$x(x+2)(x+4) = -105$
$x(x^2 + 6x + 8) = -105$
$x^3 + 6x^2 + 8x + 105 = 0$

Use factor theorem to solve.
Let $f(x) = x^3 + 6x^2 + 8x + 105$

Since the product of the three odd integers is negative, try negative values only.

Try $a = -3$ $\quad f(-3) = (-3)^3 + 6(-3)^2 + 8(-3) + 105 = 108$

Try $a = -7$ $\quad f(-7) = (-7)^3 + 6(-7)^2 + 8(-7) + 105 = 0$

```
-7  |   1    6    8    105
    |       -7    7   -105
    -----------------------------------
        1   -1   15  | 0
```

$(x+7)(x^2 - x + 15) = 0$

For $x^2 - x + 15$, use Quadratic formula

$$x = \frac{1 \pm \sqrt{(-1)^2 - 4(1)(15)}}{2(1)} = \frac{1 \pm \sqrt{-59}}{2} = \text{no real solution}$$

$\therefore x = -7$

The three consecutive odd integers are -7, -5, and -3.

Ex. Kevin is preparing to make an ice sculpture. He has a block of ice that is 3 ft wide, 4 ft high, and 5 ft long. Kevin wants to reduce the size of the block of ice by removing the same amount from each of the three dimensions. He wants to reduce the volume of the ice block to be 24 ft^3. Determine how much should Kevin remove from each dimension.

Let $x =$ amount to be reduce for each dimension

new dimensions: $l = 5 - x, w = 3 - x, h = 4 - x$

$V_{new} = 24$ $\qquad$ $V_{new} = (5 - x)(3 - x)(4 - x)$

$$(5 - x)(3 - x)(4 - x) = 24$$

$$(15 - 8x + x^2)(4 - x) = 24$$

$$-x^3 + 12x^2 - 47x + 60 = 24$$

$$x^3 - 12x^2 + 47x - 36 = 0$$

Use factor theorem to solve.

Let $f(x) = x^3 - 12x^2 + 47x - 36$

Try $a = 1$ $\qquad$ $f(1) = 1 - 12 + 47 - 36 = 0$

```
1   |   1    -12   47    -36
    |          1   -11    36
-------------------------------------
        1    -11   36   | 0
```

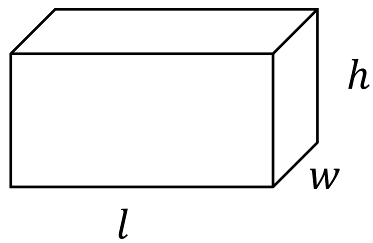
$(x - 1)(x^2 - 11x + 36) = 0$

$$x = \frac{1 \pm \sqrt{(-11)^2 - 4(1)(36)}}{2(1)} = \frac{1 \pm \sqrt{-23}}{2} = \text{no real solution}$$

$\therefore x = 1$

Kevin should remove 1 ft from each dimension.

Ex. A box is constructed such that the length is twice the width and the height is 2 cm longer than the width, with a volume of 350 cm^3. Find the dimensions of the box.



Let $w = width$

$l = 2w$ $\qquad h = w + 2$

$V = l \cdot w \cdot h$ $\qquad V = 350$

$$lwh = 350$$
$$2w \cdot w \cdot (w + 2) = 350$$
$$2w^3 + 4w^2 - 350 = 0$$
$$w^3 + 2w^2 - 175 = 0$$

Use factor theorem:

Try $a = 5$

$f(5) = 2(125) + 4(25) - 350 = 0$

$\therefore$ $(w - 5)$ is a factor

5	|	2	4	0	-350
	|		10	70	350
		2	14	70 |	0

$$(w - 5)(w^2 + 7w + 35) = 0$$

To solve $w^2 + 7w + 35$, use quadratic formula

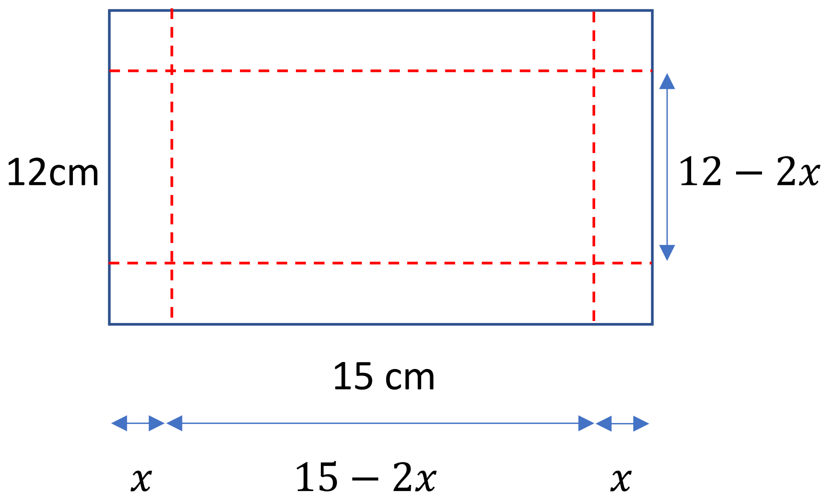
$w = \frac{-7 \pm \sqrt{7^2 - 4(1)(35)}}{2(1)} = \frac{7 \pm \sqrt{-91}}{2}$ = no real solution

$\therefore$ the only solution is $w = 5$

$l = 2w = 10$ cm $\qquad w = 5$ cm $\qquad h = w + 2 = 7$ cm

The dimensions of the rectangular box are 10 cm x 5 cm x 7 cm.

Ex. An open rectangular box is constructed by cutting a square of length x from each corner of a 12 cm by 15 cm rectangular piece of cardboard, then folding up the sides. What is the length of the square that must be cut from each corner if the volume is 112 cm^3? ($x \geq 1$)



$V = l \cdot w \cdot h$ $\qquad$ $V = 112$

$\therefore l \cdot w \cdot h = 112$

$l = 15 - 2x$

$w = 12 - 2x$

$h = x$

Sub in the expression for the length, width and height.

$$(15 - 2x)(12 - 2x)(x) = 112$$
$$x(180 - 54x + 4x^2) = 112$$
$$4x^3 - 54x^2 + 180x = 112$$
$$2x^3 - 27x^2 + 90x - 56 = 0$$

Use factor theorem:

$f(4) = 0$ $\qquad$ $a = 4$ $\qquad$ factor of $x - 4$

4	2	-27	90	-56
		8	-76	56
	2	-19	14	0

$(x - 4)(2x^2 - 19x + 14) = 0$ $\qquad$ For $2x^2 - 19x + 14$, use quadratic formula

$$x = \frac{19 \pm \sqrt{(-19)^2 - 4(2)(14)}}{2(2)} = \frac{19 \pm \sqrt{249}}{4}$$

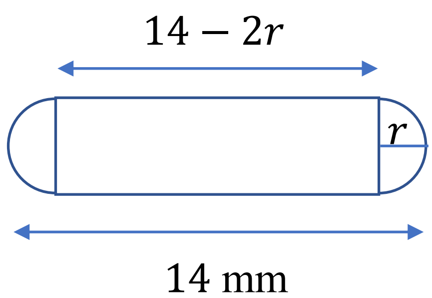
$$x = 8.69, 0.81, 4$$

Reject $x = 8.69$ because it would exceed the dimensions of the paper

Reject $x = 0.81$ because $x \geq 1$

$\therefore$ the length of the corners is 4 cm

Ex. A vitamin capsule has the shape of a right circular cylinder with hemispheres on each end. The total length of the capsule is 14 mm, and its volume is 108π mm^3. Find the radius r of the capsule.



$$V_{sphere} = \frac{4\pi r^3}{3}$$

$$V_{cylinder} = \pi r^2 h$$
$$h_{cylinder} = 14 - 2r$$

$$V_{capsule} = V_{sphere} + V_{cylinder}$$
$$V_{capsule} = \frac{4\pi r^3}{3} + \pi r^2 h$$
$$V_{capsule} = 108\pi$$

$$\frac{4\pi r^3}{3} + \pi r^2 h = 108\pi$$
$$4r^3 + 3r^2 h = 324$$

Sub in $h = 14 - 2r$, and simplify the equation

$$4r^3 + 3r^2(14 - 2r) - 324 = 0$$
$$4r^3 + 42r^2 - 6r^3 - 324 = 0$$
$$-2r^3 + 42r^2 - 324 = 0$$
$$r^3 - 21r^2 + 162 = 0$$

Use factor theorem

$$f(3) = 0$$
$\therefore a = 3$ and $r - 3$ is a factor

3	|	1	-21	0	162
	|		3	-54	-162
		1	-18	-54 |	0

$$(r - 3)(r^2 - 18r - 54) = 0$$

Since $r^2 - 18r - 54$ is not factorable, so use quadratic formula

$$r = \frac{18 \pm \sqrt{(-18)^2 - 4(1)(-54)}}{2(1)}$$

$$r = \frac{18 \pm \sqrt{540}}{2} = \frac{18 \pm 6\sqrt{15}}{2}$$

$$r = 9 + 3\sqrt{15}, 9 - 3\sqrt{15}, 3$$

$$r = 9 - 3\sqrt{15} \approx -3$$

$$r = 9 + 3\sqrt{15} \approx 20.6$$

Reject $r = 9 - 3\sqrt{15}$
because it is negative; radius cannot be negative

Reject $r = 9 + 3\sqrt{15}$
because radius would exceed the length of the capsule

$\therefore$ the length of the radius is 3 mm

3.5 Homework

1, 3, 6, 7, 9, 11, 12