44. $\sin A = \frac{-3}{5}$, therefore, by Pythagoras Theorem, $\cos A = \frac{4}{5}$ in quadrant IV

$\cos B = \frac{3}{5}$, therefore, by Pythagoras Theorem, $\sin B = \frac{-4}{5}$ in quadrant IV

$\sin(A+B) = \sin A \cdot \cos B + \cos A \cdot \sin B = \left(\frac{-3}{5}\right)\left(\frac{3}{5}\right) + \left(\frac{4}{5}\right)\left(\frac{-4}{5}\right) = \frac{-9}{25} - \frac{16}{25} = -1$. Answer is a.

45. Tangent has a value of $-\sqrt{3}$ at $\frac{2\pi}{3}$ and $\frac{5\pi}{3}$ which is π distance apart.

Therefore, $bx = \frac{2\pi}{3} + n\pi \rightarrow x = \frac{2\pi}{3b} + \frac{n\pi}{b}$. Answer is b.

46.
$$\cos\left(x+\frac{\pi}{2}\right) - \cos\left(x-\frac{\pi}{2}\right) = \left(\cos x\cos\frac{\pi}{2} - \sin x \sin\frac{\pi}{2}\right) - \left(\cos x\cos\frac{\pi}{2} + \sin x\sin\frac{\pi}{2}\right)$$
$$= (0 - \sin x) - (0 + \sin x)$$
$$= -2\sin x.$$ Answer is b.

47. $8\sin^4 x + 2\sin^2 x - 1 = 0 \rightarrow (4\sin^2 x - 1)(2\sin^2 x + 1) = 0 \rightarrow \sin x = \pm\frac{1}{2}$, $\sin x = \pm\sqrt{\frac{-1}{2}}$ (reject)

$\sin x = \pm\frac{1}{2}$ at $30°$, $150°$, $210°$, $330°$, $\sin x = \pm\sqrt{\frac{-1}{2}}$, ϕ. Answer is c.

48.
$$(\sin x - \cos x)^2 - (\sin x + \cos x)^2 = (\sin^2 x - 2\sin x\cos x + \cos^2 x) - (\sin x^2 + 2\sin x\cos x + \cos^2 x)$$
$$= 1 - 2\sin x\cos x - 1 - 2\sin x \cos x = -4\sin x\cos x = -2\sin 2x.$$

Answer is d.

49. cosine has value $-\frac{1}{2}$ at $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$, therefore, $2x = \frac{2\pi}{3} + 2n\pi$, $2x = \frac{4\pi}{3} + 2n\pi$

$x = \frac{\pi}{3} + n\pi$, $x = \frac{2\pi}{3} + n\pi$ Answer is b.

50. $\cos x + 2\cos^2 x = 0 \rightarrow \cos x(1 + 2\cos x) = 0 \rightarrow \cos x = 0, -\frac{1}{2}$

$\cos x = 0$ at $\frac{\pi}{2} + n\pi$, $\cos x = -\frac{1}{2}$ at $\frac{2\pi}{3} + 2n\pi$ and $\frac{4\pi}{3} + 2n\pi$. Answer is b.

Combinatorics Solutions

7.1 Exercise Set

1. a) 3 ways from Calgary to Vancouver;
2 ways from Vancouver to Victoria.
Therefore, $3 \times 2 = 6$ ways altogether

b) car–boat, car–plane, train–boat, train–plane, bus–boat, bus–plane

2. a) $4 \times 3 = 12$ ways from A to C via B

b) $\underbrace{4 \times 3}_{\text{going}} \times \underbrace{3 \times 4}_{\text{return}} = 144$ ways from A to C and back to A (via B both ways).

c) $\underbrace{4 \times 3}_{\text{going}} \times \underbrace{2 \times 3}_{\text{return – one less route on return trip}} = 72$

3. $4 \times 4 \times 4 \times 4 \times 4 = 4^5 = 1024$ ways

4. TT, TH, H1, H2 H3, H4, H5, H6 = 8 possibilities

5. $6 \times 4 \times 5 = 120$ different ways

6. Horses 1, 4, 6 can come in $3 \cdot 2 \cdot 1 = 6$ ways
There are 4 non-winning horses. They can come $4 \cdot 3 \cdot 2 \cdot 1 = 24$ ways.
Therefore, $6 \times 24 = 144$ different ways in which the horse race can end if horses 1, 4, 6 take the first 3 place

7. Juice – 3 choices, Toast – 2 choices, Eggs – 3 choices, Beverage – 3 choices
Therefore, $3 \times 2 \times 3 \times 3 = 54$ breakfast combinations are possible.

8. $4 \times 3 \times 2 = 24$ ways

9. a) Each letter has 26 choices, and each digit 10 choices.
$26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 = 17\,576\,000$ license plates

b) The first letter has 26 choices, the second letter has 25 choices, and the third letter has 24 choices. The first digit has 10 choices, the second digit has only 9 choices, and the third digit has just 8 choices.
$26 \cdot 25 \cdot 24 \cdot 10 \cdot 9 \cdot 8 = 11\,232\,000$ license plates

c) There are 26 license plates with all letters the same, and 10 digits with all three numbers the same. You can only use a digit twice and a number twice, so you have to subtract the possibilities that use the letters and digits three times.
$(26 \cdot 26 \cdot 26 - 26) \cdot (10 \cdot 10 \cdot 10 - 10) = 17\,374\,500$ license plates

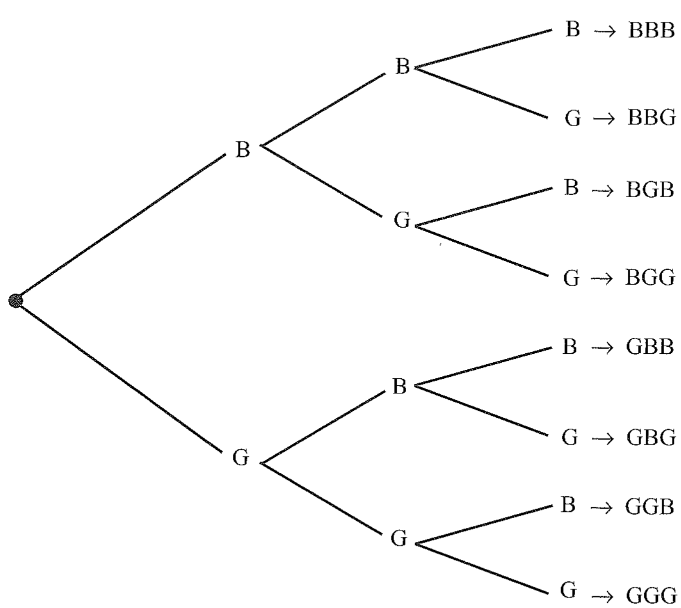
d) $(26 \cdot 26 \cdot 26 - 26) \cdot (10 \cdot 9 \cdot 8) = 12\,636\,000$ license plates

10. There are $26 \cdot 25 \cdot 24$ three-letter words in which all letters are different. Each of these three-letter words can be arranged in six ways.

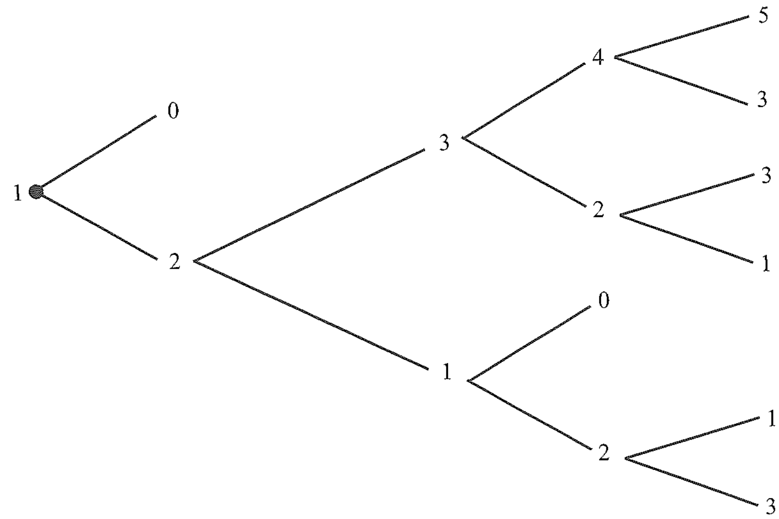
Example: ABC → ABC, ACB, BAC, BCA, CAB, CBA Only 1 out of 6 is in alphabetical order

Therefore, $\dfrac{26 \cdot 25 \cdot 24}{6} = 2\,600$ three-letter words in alphabetical order.

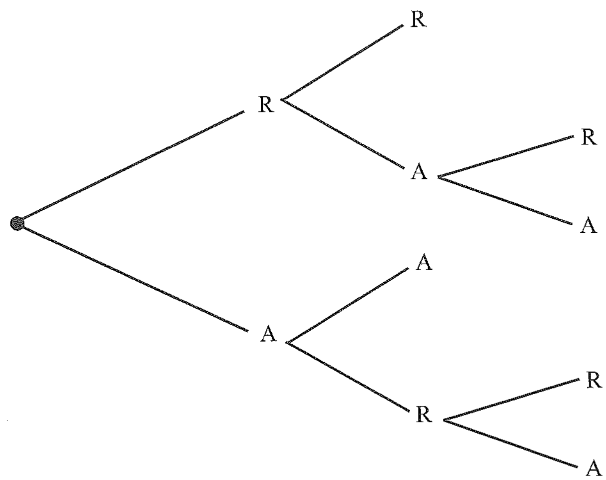
11. Eight possible outcomes.



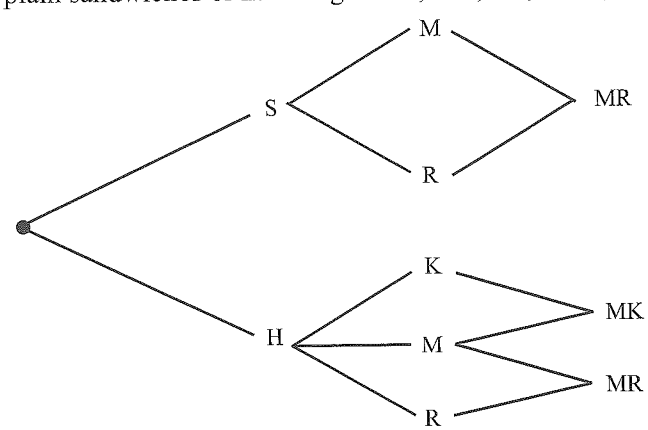
12. Eight possible outcomes.



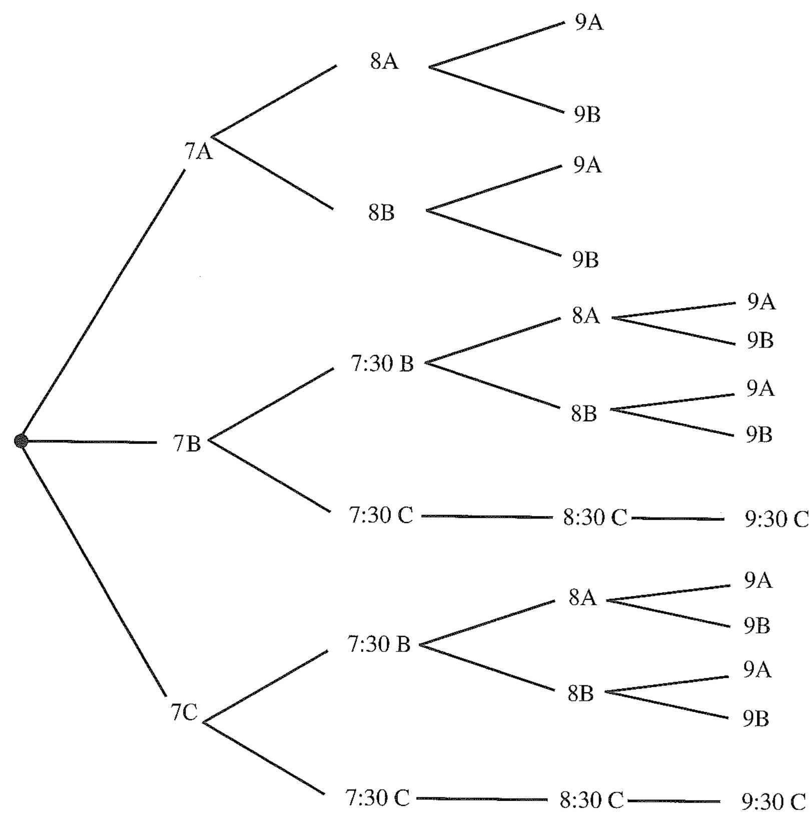
13. Six possible outcomes.



14. Ten outcomes, counting plain sandwiches or hamburgers: S, SM, SR, SMR, H, HK, HM, HR, HMK, HMR



15. Fourteen different combinations of complete shows.



16. a) $\frac{200!}{198!} = \frac{200 \times 199 \times 198!}{198!} = 200 \times 199 = 39\,800$

b) $\frac{100! - 98!}{98!} = \frac{100 \times 99 \times 98! - 98!}{98!} = 100 \times 99 - 1 = 9899$

c) $\frac{100!\ 98!}{99!\ 97!} = \frac{100 \times 99! \times 98 \times 97!}{99! \times 97!} = 100 \times 98 = 9800$

17. a) $20 \cdot 19 \cdot 18 \cdot 17 = \frac{20!}{16!}$

b) $6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$ ways

c) $7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5040$ different 7-digit numbers

d) $\frac{7!}{3!} = \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3!}{3!} = 7 \cdot 6 \cdot 5 \cdot 4 = 840$ arrangements

18. a) $\frac{(n-1)!(n+1)!}{(n!)^2} = \frac{(n-1)!(n+1)n!}{n! \cdot n(n-1)!} = \frac{n+1}{n}$

b) $\frac{(2n-1)!(n+1)!}{(2n+1)!(n-1)!} = \frac{(2n-1)!(n+1)(n)(n-1)!}{(2n+1)(2n)(2n-1)!(n-1)!} = \frac{n(n+1)}{2n(2n+1)} = \frac{n+1}{2(2n+1)}$

18. c) $\dfrac{(n-1)!-(n-2)!}{(n-1)!}=\dfrac{(n-1)(n-2)!-(n-2)!}{(n-1)(n-2)!}=\dfrac{(n-1)-1}{n-1}=\dfrac{n-2}{n-1}$

d) $\dfrac{(n-1)!}{n!}+\dfrac{(n-2)!}{n!}=\dfrac{(n-1)(n-2)!+(n-2)!}{n(n-1)(n-2)!}=\dfrac{(n-1)+1}{n(n-1)}=\dfrac{n}{n(n-1)}=\dfrac{1}{n-1}$

e) $\dfrac{n!-(n-1)!}{(n+1)!-2(n-1)!}=\dfrac{n(n-1)!-(n-1)!}{(n+1)(n)(n-1)!-2(n-1)!}=\dfrac{(n-1)}{n(n+1)-2}=\dfrac{n-1}{n^2+n-2}$

$\dfrac{n-1}{(n-1)(n+2)}=\dfrac{1}{n+2}$

f) $\dfrac{n!-6(n-2)!}{(n-3)(n-2)!}=\dfrac{n(n-1)(n-2)!-6(n-2)!}{(n-3)(n-2)!}=\dfrac{n(n-1)-6}{(n-3)}=\dfrac{n^2-n-6}{n-3}=\dfrac{(n-3)(n+2)}{(n-3)}=n+2$

19. a) $\dfrac{3!(n-1)!}{(n-3)!}=72\rightarrow\dfrac{6(n-1)(n-2)(n-3)!}{(n-3)!}=72\rightarrow(n-1)(n-2)=12\rightarrow n^2-3n+2=12$

$n^2-3n-10=0\rightarrow(n-5)\ (n+2)=0\rightarrow n-5=0$ or $n+2=0\rightarrow n=5$, reject -2

b) $\dfrac{(2n-1)!}{2!(2n-3)!}=10\rightarrow\dfrac{(2n-1)(2n-2)(2n-3)!}{2(2n-3)!}=10\rightarrow\dfrac{2(2n-1)(n-1)}{2}=10\rightarrow 2n^2-3n+1=10$

$2n^2-3n-9=0\rightarrow(2n+3)(n-3)=0\rightarrow 2n+3=0$ or $n-3=0\rightarrow n=3$, reject $\dfrac{-3}{2}$

7.2 Exercise Set

1. $P(40,5)=\dfrac{40!}{(40-5)!}=\dfrac{40!}{35!}$ = 78 960 960 possible ticket selections.

2. ${}_{120}P_6=\dfrac{120!}{(120-6)!}$ = $120\cdot119\cdot118\cdot117\cdot116\cdot115=2.63\times10^{12}$ possible room assignments.

3. 9! = 362 880 possible batting orders.

4. ${}_6P_4=\dfrac{6!}{(6-4)!}$ = 360 possible groups of 4 letters.

5. a) 6! = 720 possible seating arrangements.

b) We can sit the boys and girls in 2 ways: BBBGGG or GGGBBB.
The boys can sit 3! ways and the girls 3! ways.
Therefore, 3! × 3! × 2 ways = 72 ways.

c) We can sit the boys together in 4 ways: BBBGGG, GBBBGG, GGBBBG, GGGBBB.
The boys can sit 3! and the girls 3!
Therefore, 3! × 3! × 4 ways = 144 ways.

5. d) Alternate the boys and girls can sit two ways: BGBGBG or GBGBGB
The boys can sit $P(3,3)$ and the girls $P(3,3)$
Therefore, $P(3,3)\times P(3,3)\times 2 = 3!\times 3!\times 2 = 72$ ways

6. The two people in the front can sit $P(2,1) = 2$ ways
The four that sit behind can sit $P(4,4) = 24$ ways
Therefore, $2\times 24 = 48$ ways

7. The delivery man can travel $6!$ ways. Since opposite routes, e.g., ABCDEF and FEDCBA, count as the same route, there are $\frac{6!}{2} = 360$ different routes.

8. If there are no restrictions, then 5 people can be seated in 5! ways. A is to the left of B for half of these ways. Therefore, $\frac{5!}{2} = 60$ ways with A to the left of B.

9. If there are no restrictions, then 5 people can be seated in $5!$ ways. If AB sat next to each other, consider them one person, so they could do this $4!$ ways. But they can also switch positions BA, which also has $4!$ ways. Therefore, they cannot sit together in $5! - 2 \times 4! = 72$ ways.

10. Method 1:

The first couple has 10 choices of seats in which to sit.
The second couple has 8 choices of seats in which to sit.
The third couple has 6 choices of seats in which to sit.
The fourth couple has 4 choices of seats in which to sit.
The fifth couple has 2 choices of seats in which to sit.
Therefore, the couples can sit $10 \cdot 8 \cdot 6 \cdot 4 \cdot 2 = 3\,840$ ways.

Method 2:

The 5 couples can sit together in $5!$ ways, e.g., Aa Bb Cc Dd Ee. Each couple can switch, e.g., aA, bB; there are 2^5 ways of this happening. Therefore, the couples can sit $5! \cdot 2^5 = 3\,840$ ways.

11. One person can sit at any position around the table. The other 4 can arrange themselves in $4!$ ways $= 24$ ways

12. $\frac{10!}{3!\,3!\,2!} = 50\,400$ ten-letter "words."

13. $\frac{10!}{5!\,3!\,2!} = 2\,520$ ways.

14. $\frac{9!}{2!\,3!\,4!} = 1\,260$ ways.

15. $\frac{10!}{2!\,3!\,3!\,2!} = 25\,200$ different ten-letter codes.

16. $\frac{12!}{5!\,4!\,3!} = 27\,720$ ways.

17. There are $30!$ ways of arranging 30 teachers. Each school has 5 teachers, who have $5!$ ways of being arranged.

$$\frac{30!}{(5!)^6} = 8.88 \times 10^{19} \text{ ways.}$$

18. a) $\frac{9!}{2!\,2!} = 90\,720$ arrangements.

b) If the first letter is a "C," then there are only 8 letters left, including two "Os."

Therefore, $1\times\frac{8!}{2!} = 20\,160$ arrangements.

c) First letter an "O" then 8 letters left with two "Cs", thus $1\times\frac{8!}{2!}$

First letter not an "O" then 5 choices for first letter with 8 letters left with two "Os" and two "Cs",

thus $5\times\frac{8!}{2!2!}$. Total is $1\times\frac{8!}{2!}+5\times\frac{8!}{2!2!}=70\,560$ ways.

or answer 18a) – 18b) = 90 720 – 20 160 = 70 560 ways

d) Consider the two "Cs" as one letter; there are 8 letters, including two "Os."

Therefore, $\frac{8!}{2!} = 20\,160$

19. $\frac{25!}{12!\cdot 3!\cdot 10!} = 1\,487\,285\,800$ ways of selecting the jury.

20. 4 blocks + 5 blocks + 7 blocks + 4 blocks = 20 blocks
4 blocks east + 4 blocks east = 8 blocks east
5 blocks north + 7 blocks north = 12 blocks north

Therefore, $\frac{20!}{8!\,12!} = 125\,970$ different routes.

21. If everyone gets along, there are $\frac{6!}{2!\,2!\,2!} = 90$ ways. But 2 people cannot work together in any of the 3 offices. So, consider the 3 offices as 1, 2, 3, and the workers as A, B, C, D, E and F. When A and B are in office 1, then office 2 and 3 can have workers:

C, D	E, F
C, E	D, F
C, F	D, E
E, F	C, D
D, F	C, E
D, E	C, F

for six arrangements. The same combinations of CDEF are possible when A and B are in office 2 or 3, for a total of 6 × 3 = 18 ways that don't work. Therefore, 90 – 18 = 72 possible working situations.

7.3 Exercise Set

1. a) Changing the order of the 5 digits results in a different number; use permutation.

b) The order of selecting the marbles is not important; use combination.

c) A combination lock of numbers 20–30–15 and 30–15–20 are different combinations so order is important; use permutation.

d) The order you place each team is not important; use combination.

e) There is no difference in the selection of the 3 people; use combination.

f) The three positions are different, so order matters; use permutation.

2. Order is not important, so this is a combination.

$${}_7C_3 = \frac{7!}{3!(7-3)!} = 35 \text{ ways.}$$

3. Order is not important.

$$C(20,\ 3) = \frac{20!}{3!(20-3)!} = 1\ 140 \text{ ways.}$$

4. Total people equals 3 + 12 = 15. Order is not important.

$$\binom{15}{5} = \frac{15!}{5!(15-5)!} = 3\ 003 \text{ ways.}$$

5. Total number of coins is 6. Order is not important.

$${}_6C_3 = \frac{6!}{3!(6-3)!} = 20 \text{ different sums of money.}$$

6. If the couple does not attend, there are 8 students left.

So $${}_8C_4 = \frac{8!}{4!(8-4)!} = 70 \text{ ways.}$$

If the couple is selected, then only 2 can come from the other 8 students.

So $${}_8C_2 = \frac{8!}{2!(8-2)!} = 28 \text{ ways.} \qquad \text{Total} = 70 + 28 = 98 \text{ ways.}$$

7. A card deck consists of 13 hearts, 13 diamonds, 13 clubs, and 13 spades.

So $\binom{13}{3}$ for the 3 hearts and $\binom{13}{2}$ for the 2 clubs. Therefore, $\binom{13}{3} \times \binom{13}{2} = 22\ 308$ card hands.

8. a) Order is not important. A chord needs 2 points on the circumference.

So $$C(10,\ 2) = \frac{10!}{2!(10-2)!} = 45 \text{ chords.}$$

b) A triangle has 3 points. So $${}_{10}C_3 = \frac{10!}{3!(10-3)!} = 120 \text{ triangles.}$$

c) A quadrilateral has 4 points. So $$\binom{10}{4} = \frac{10!}{4!(10-4)!} = 210 \text{ quadrilaterals.}$$

d) General formula $${}_nC_r = \frac{n!}{r!(n-r)!}$$

9. a) If no restrictions, then 12 students try to get 4 jobs.

So $${}_{12}C_4 = \frac{12!}{4!(12-4)!} = 495 \text{ ways.}$$

9. b) 2 men out of 5 is $C(5, 2)$, 2 women out of 7 is $C(7, 2)$.

So $C(5, 2) \times C(7, 2) = 10 \times 21 = 210$ ways.

c) At least 2 women means: 2 women + 2 men, or 3 women + 1 man, or 4 women and no men.

So $\binom{7}{2}\binom{5}{2}+\binom{7}{3}\binom{5}{1}+\binom{7}{4}\binom{5}{0} = 21\times 10 + 35\times 5 + 35\times 1 = 420$ ways.

10. a) If no restrictions, then select 6 bulbs from 30. Therefore, ${}_{30}C_6 = \dfrac{30!}{6!(30-6)!} = 593\ 775$ ways.

b) If there are 30 bulbs with 5 defective ones, then there are 25 non-defective bulbs.

$C(25, 6) = \dfrac{25!}{6!(25-6)!} = 177\ 100$ ways.

c) If we must have 2 defective ones, then there are 4 non-defective bulbs selected.

$\binom{5}{2} \times \binom{25}{4} = \dfrac{5!}{2!(5-2)!} \times \dfrac{25!}{4!(25-4)!} = 10\times 12\ 650 = 126\ 500$ ways.

11. a) A straight flush consists of 1, 2, 3, 4, 5, to 10, J, Q, K, 1 of the same suit, or 10×4 suits $= 40$ hands.

b) For 4 of a kind, out of 4 cards, you want all 4, so ${}_4C_4$, and this can be done for 13 numbers.

The fifth card is 1 out of the 48 cards remaining.

Therefore, $\binom{4}{4}\binom{13}{1}\binom{48}{1} = 624$ possible four-of-a-kind hands in a 5-card hand.

c) For 3 of a kind $\binom{4}{3}$, with 13 different choices $\binom{13}{1}$.

For a pair $\binom{4}{2}$, with 12 different choices $\binom{12}{1}$ (one number is used up with the 3 of a kind).

Therefore, $\underbrace{\binom{4}{3}\binom{13}{1}}_{\text{3 of a kind}}\underbrace{\binom{4}{2}\binom{12}{1}}_{\text{a pair}} = 3\ 744$ different full house possibilities in a 5-card hand.

d) A flush consists of 5 cards all of the same suit $\binom{13}{5}$ and there are 4 suits. We have to subtract the straight flushes, which is $4 \times 10 = 40$.

Therefore, $4\cdot\binom{13}{5} - 40 = 5\ 108$ possible flushes in a 5-card hand.

e) There are 10 different straights from (1, 2, 3, 4, 5) up to (10, J, Q, K, 1); each of the 5 cards can be a club, diamond, heart or spade.

Therefore, $10\cdot\left({}_4C_1\right)^5$, but this includes card hands that have all 5 cards of the same suit (straight flushes) so subtract 10 different card hands times the 4 suits $= 40$

$10\cdot\left({}_4C_1\right)^5 - 40 = 10\ 200$ possible straights in a 5-card hand.

11. f) For 3 of a kind $\binom{4}{3}$, for two single cards $\binom{4}{1}^2$, there are 13 choices for 3 of a kind, then 12, then 11, for the 2 single cards, but order does not matter for the 2 single cards so divide by $2!$. Therefore, $\binom{4}{3}\binom{4}{1}^2 \cdot \frac{13\cdot 12\cdot 11}{2!} = 54\ 912$, Method 2: for 3 of a kind $\binom{4}{3}$, of 13 suits you want one $\binom{13}{1}$, for 2 single cards $\binom{4}{1}^2$, of 12 suits left you want 2 so $\binom{12}{2}$, therefore $\binom{4}{3}\binom{13}{1}\binom{4}{1}^2\binom{12}{2} = 54\ 912$ possible 5-card hands with 3 of a kind.

g) For 2 pairs, $\binom{4}{2}^2$, for the 5th card $\binom{4}{1}$, there are 13 choices for the first pair, 12 choices for next pair and 11 choices for the single card, but order for the two pairs does not matter so divide by $2!$.

Therefore $\binom{4}{2}^2 \cdot \binom{4}{1} \cdot \frac{13\cdot 12\cdot 11}{2!} = 123\ 552$, Method 2: for 2 pair $\binom{4}{2}^2$, of 13 suits you want two $\binom{13}{2}$, for asingle card $\binom{4}{1}$, of 11 suits left you want one $\binom{11}{1}$, therefore $\binom{4}{2}^2\binom{13}{2}\binom{4}{1}\binom{11}{1} = 123\ 552$ different 5-card hands with 2 pair.

h) For a pair $\binom{4}{2}$, for 3 single cards $\binom{4}{1}^3$, you have 13 choices for the pair, then $12\cdot 11\cdot 10$ for the 3 single cards, but order does not matter, so divide by $3!$. Therefore, $\binom{4}{2}\binom{4}{1}^3 \cdot \frac{13\cdot 12\cdot 11\cdot 10}{3!} = 1\ 098\ 240$, Method 2: for a pair $\binom{4}{2}$, of 13 suits you want one $\binom{13}{1}$, for 3 single cards $\binom{4}{1}^3$, of 12 suits left you want 3 of them $\binom{12}{3}$, therefore $\binom{4}{2}\binom{13}{1}\binom{4}{1}^3\binom{12}{3} = 1\ 098\ 240$ different 5-card hands with a single pair.

i) Of 13 different cards, you want 5 of them $\binom{13}{5}$, with 4 suits for each different card pick $\binom{4}{1}^5$, but we must subtract straights, flushes, and straight flushes, so $\binom{13}{5}\binom{4}{1}^5 - 10\ 200 - 5\ 108 - 40 = 1\ 302\ 540$ or 52 choices for the first card pick then 48 for the next, then 44, then 40, then 36. But order does not matter, so divide by $5!$, and subtract straights, flushes, and straight flushes, so $\frac{52\cdot 48\cdot 44\cdot 40\cdot 36}{5!} - 10\ 200 - 5\ 108 - 40 = 1\ 302\ 540$ different 5-card hands with a high card.

j) 5-card hand $\binom{52}{5} = 2\ 598\ 960$. Sum of different 5-card hands are $40 + 624 + 3\ 744 + 5\ 108 + 10\ 200 + 54\ 912 + 123\ 552 + 1\ 098\ 240 + 1\ 302\ 540 = 2\ 598\ 960$

7.4 Exercise Set

1. Method 1 (Pascal's Triangle)

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

$$(x+y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$$

Method 2 (binomial expansion)

$$(x+y)^5 = \binom{5}{0}x^5y^0 + \binom{5}{1}x^4y^1 + \binom{5}{2}x^3y^2 + \binom{5}{3}x^2y^3 + \binom{5}{4}xy^4 + \binom{5}{5}x^0y^5$$

$$= x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$$

2. Method 1 (Pascal's Triangle)

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

$$(x^2-2y)^6 = 1(x^2)^6 + 6(x^2)^5(-2y)^1 + 15(x^2)^4(-2y)^2 + 20(x^2)^3(-2y)^3 +$$

$$15(x^2)^2(-2y)^4 + 6(x^2)^1(-2y)^5 + 1(x^2)^0(-2y)^6$$

$$= x^{12} - 12x^{10}y + 60x^8y^2 - 160x^6y^3 + 240x^4y^4 - 192x^2y^5 + 64y^6$$

Method 2 (binomial expansion)

$$(x^2-2y)^6 = \binom{6}{0}(x^2)^6 + \binom{6}{1}(x^2)^5(-2y)^1 + \binom{6}{2}(x^2)^4(-2y)^2 + \binom{6}{3}(x^2)^3(-2y)^3 +$$

$$\binom{6}{4}(x^2)^2(-2y)^4 + \binom{6}{5}(x^2)^1(-2y)^5 + \binom{6}{6}(x^2)^0(-2y)^6$$

$$= x^{12} - 12x^{10}y + 60x^8y^2 - 160x^6y^3 + 240x^4y^4 - 192x^2y^5 + 64y^6$$

3. Method 1 (Pascal's Triangle)

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

$$\left(2x - \frac{1}{y^2}\right)^5 = 1(2x)^5 + 5(2x)^4\left(-\frac{1}{y^2}\right)^1 + 10(2x)^3\left(-\frac{1}{y^2}\right)^2 +$$

$$10(2x)^2\left(-\frac{1}{y^2}\right)^3 + 5(2x)^1\left(-\frac{1}{y^2}\right)^4 + 1\,(2x)^0\left(-\frac{1}{y^2}\right)^5$$

$$= 32x^5 - \frac{80x^4}{y^2} + \frac{80x^3}{y^4} - \frac{40x^2}{y^6} + \frac{10x}{y^8} - \frac{1}{y^{10}}$$

Method 2 (binomial expansion)

$$\left(2x - \frac{1}{y^2}\right)^5 = \binom{5}{0}(2x)^5 + \binom{5}{1}(2x)^4\left(-\frac{1}{y^2}\right)^1 + \binom{5}{2}(2x)^3\left(-\frac{1}{y^2}\right)^2 + \binom{5}{3}(2x)^2\left(-\frac{1}{y^2}\right)^3 +$$

$$\binom{5}{4}(2x)^1\left(-\frac{1}{y^2}\right)^4 + \binom{5}{5}(2x)^0\left(-\frac{1}{y^2}\right)^5$$

$$= 32x^5 - \frac{80x^4}{y^2} + \frac{80x^3}{y^4} - \frac{40x^2}{y^6} + \frac{10x}{y^8} - \frac{1}{y^{10}}$$

4. Method 1 (Pascal's Triangle)

$$\begin{array}{c} 1 \\ 1 \quad 1 \\ 1 \quad 2 \quad 1 \\ 1 \quad 3 \quad 3 \quad 1 \\ \boxed{1 \quad 4 \quad 6 \quad 4 \quad 1} \end{array}$$

$$\left(3x^2-\frac{1}{2y}\right)^4 = 1\left(3x^2\right)^4+4\left(3x^2\right)^3\left(-\frac{1}{2y}\right)^1+6\left(3x^2\right)^2\left(-\frac{1}{2y}\right)^2+$$
$$4\left(3x^2\right)^1\left(-\frac{1}{2y}\right)^3+1\left(3x^2\right)^0\left(-\frac{1}{2y}\right)^4 = 81x^8-\frac{54x^6}{y}+\frac{27x^4}{2y^2}-\frac{3x^2}{2y^3}+\frac{1}{16y^4}$$

Method 2 (binomial expansion)

$$\left(3x^2-\frac{1}{2y}\right)^4 = \binom{4}{0}\left(3x^2\right)^4+\binom{4}{1}\left(3x^2\right)^3\left(-\frac{1}{2y}\right)+\binom{4}{2}\left(3x^2\right)^2\left(-\frac{1}{2y}\right)^2+\binom{4}{3}\left(3x^2\right)^1\left(-\frac{1}{2y}\right)^3+$$
$$\binom{4}{4}\left(3x^2\right)^0\left(-\frac{1}{2y}\right)^4 = 81x^8-\frac{54x^6}{y}+\frac{27x^4}{2y^2}-\frac{3x^2}{2y^3}+\frac{1}{16y^4}$$

5. $t_{k+1} = {}_nC_k x^{n-k}\cdot y^k$

$t_6 = {}_9C_5(x^{9-5})(-2y)^5 = 126(x^4)(-32y^5) = -4032x^4y^5$

6. $t_{k+1} = {}_nC_k x^{n-k}\cdot y^k$

$$t_{11} = {}_{15}C_{10}\left(3x\right)^{15-10}\left(-\frac{1}{2y^2}\right)^{10} = 3003\left(3x\right)^5\left(-\frac{1}{2y^2}\right)^{10} = \frac{3003\cdot 243x^5}{1024y^{20}} = \frac{729729x^5}{1024y^{20}}$$

7. Remember, $\left(x^2-\frac{1}{y}\right)^7$ has 8 terms, so the next-to-last term is the 7th term.

$t_{k+1} = {}_nC_k x^{n-k}\cdot y^k$

$t_7 = {}_7C_6(x^2)^{7-6}\cdot(-\frac{1}{y})^6 = 7\cdot x^2\cdot\frac{1}{y^6} = \frac{7x^2}{y^6}$

8. Remember, $(3a+2b^2)^6$ has 7 terms so the middle term is the 4th term.

$t_{k+1} = {}_nC_k x^{n-k}\cdot y^k$

$t_4 = {}_6C_3(3a)^{6-3}\cdot(2b^2)^3 = 20(3a)^3\cdot(2b^2)^3 = 20\cdot 27\cdot 8\cdot a^3b^6 = 4320a^3b^6$

9. $t_{k+1} = {}_nC_k x^{n-k}\cdot y^k$

We want $x^{n-k}\cdot y^k = 1$, i.e. the resulting exponent must be zero.

So $\left(x^2\right)^{6-k}\cdot\left(\frac{1}{x}\right)^k = x^{12-2k}\cdot\left(x\right)^{-k} = x^{12-2k-k} = 1 \rightarrow x^{12-3k} = x^0$

Equating exponents, $t_{k+1} = 12-3k = 0 \rightarrow 3k = 12 \rightarrow k = 4$, therefore, $t_{4+1} = t_5$, the 5th term is

${}_6C_4\left(x^2\right)^2\left(\frac{1}{x}\right)^4 = 15$, a constant.

10. $t_{k+1} = {}_nC_k x^{n-k} \cdot y^k$

We want $x^{n-k} \cdot y^k = 1$, i.e. the resulting exponent must be zero.

So $\left(2x^3\right)^{10-k} \cdot \left(\frac{1}{x^2}\right)^k = x^{30-3k} \cdot x^{-2k} = 1 \rightarrow x^{30-3k-2k} = x^0$

Equating exponents, $t_{k+1} = 30 - 3k - 2k = 0 \rightarrow 5k = 30 \rightarrow k = 6$, therefore, $t_{6+1} = t_7$, the 7th term.

$$t_7 = {}_{10}C_6\left(2x^3\right)^4\left(-\frac{1}{x^2}\right)^6 = {}_{10}C_6 \cdot 2^4 = 3360$$

11. $t_{k+1} = {}_nC_k x^{n-k} \cdot y^k$

If the 4th term of $\left(x - \frac{1}{2}\right)^n$ is $-15x^7$ then $t_4 = \binom{n}{3} x^{n-3}\left(-\frac{1}{2}\right)^3 = -15x^7$

Equating exponents of x, $n - 3 = 7 \rightarrow n = 10$, check: $t_4 = \binom{n}{3} x^7\left(-\frac{1}{2}\right)^3 = 120x^7\left(-\frac{1}{8}\right) = -15x^7$

12. $t_{k+1} = {}_nC_k x^{n-k} \cdot y^k$

If the 7th term of $(2x-1)^n$ is $112x^2$ then ${}_nC_6 \cdot (2x)^{n-6}(-1)^6 = 112x^2 \rightarrow {}_nC_6 \cdot (2x)^{n-6} = 112x^2$

Equating exponents of x, $n - 6 = 2 \rightarrow n = 8$, check: $t_7 = \binom{8}{6}(2x)^2(-1)^6 = 28\left(4x^2\right) = 112x^2$

13. The 4th term has x^7, therefore, $t_4 = {}_{10}C_3 \cdot x^7 b^3 = 1875x^7$

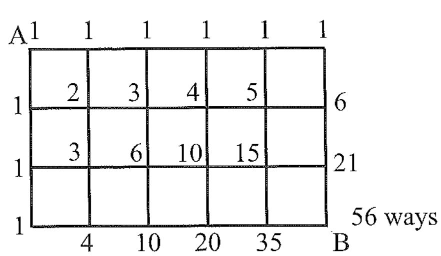
$$120b^3 = 1875$$
$$b^3 = 15.625$$
$$b = 15.625^{\frac{1}{3}}$$
$$b = 2.5$$

14. The 3rd term has x^5, therefore, $t_3 = {}_7C_2 \cdot (2x)^5 b^2 = 1512x^5$

$$21 \cdot (2x)^5 b^2 = 1512x^5$$
$$672b^2 = 1512$$
$$b^2 = 2.25$$
$$b = \sqrt{2.25}$$
$$b = 1.5$$

7.5 Exercise Set

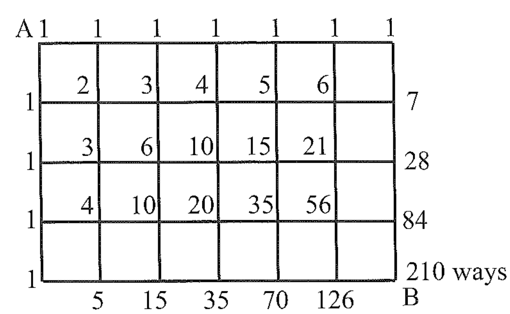
1. a)



Solution 2: 3 rows and 5 columns

$$\frac{8!}{5!3!} = 56 \text{ ways}$$

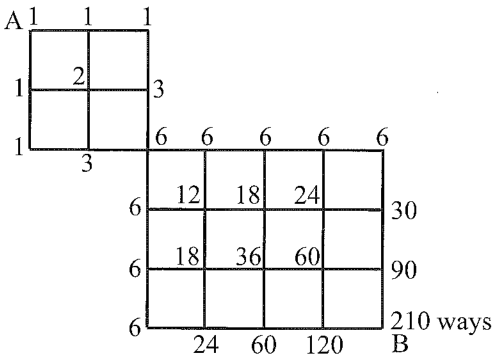
b)



Solution 2: 4 rows and 6 columns

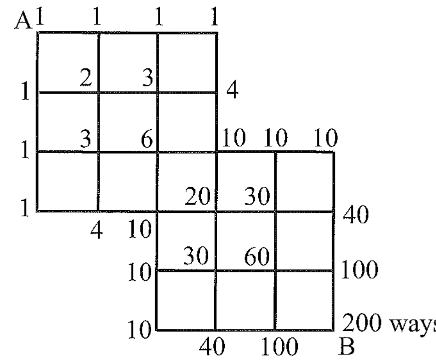
$$\frac{10!}{6!4!} = 210 \text{ ways}$$

1. c)



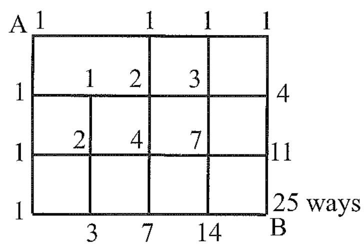
Solution 2: $\frac{4!}{2!2!} \times \frac{7!}{3!4!} = 210$ ways

d)

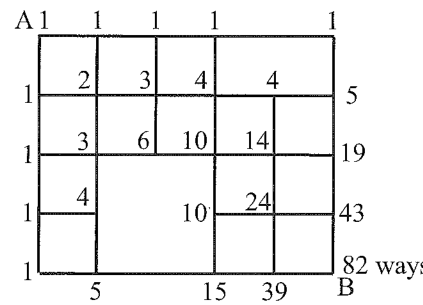


Solution 2: $\frac{6!}{3!3!} \times \frac{6!}{3!3!} \div \frac{2!}{1!1!} = 200$ ways

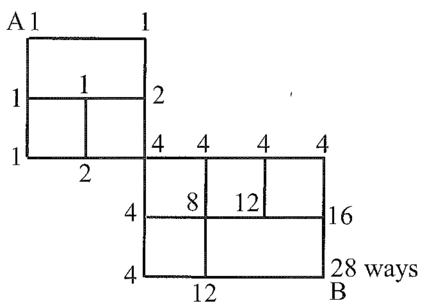
e)



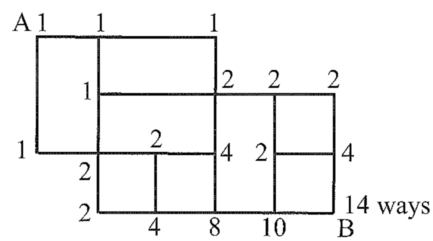
f)



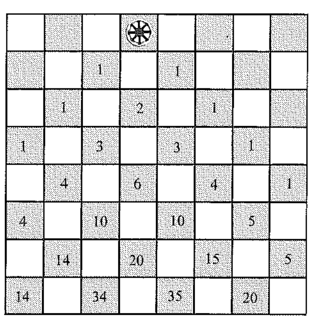
g)



h)

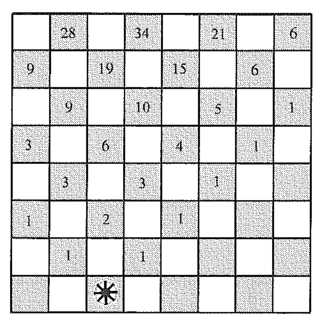


2. a)



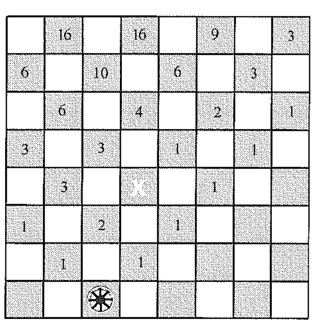
= 103 ways

b)



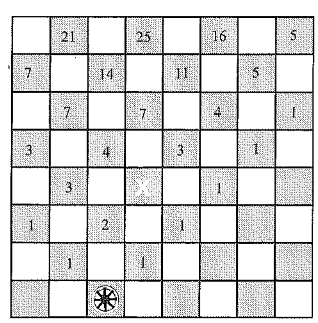
= 89 ways

2. c)



= 44 ways

d)



= 67 ways

7.6 Exercise Set

Combinations – Multiple-choice Answers

1.	c	**7.**	d	**13.**	d	**19.**	c	**25.**	a	**31.**	c	**37.**	b	**43.**	d	**49.**	a	**55.**	c
2.	c	**8.**	c	**14.**	c	**20.**	c	**26.**	d	**32.**	a	**38.**	b	**44.**	b	**50.**	b	**56.**	c
3.	b	**9.**	d	**15.**	a	**21.**	a	**27.**	b	**33.**	d	**39.**	d	**45.**	d	**51.**	d	**57.**	b
4.	b	**10.**	d	**16.**	c	**22.**	b	**28.**	a	**34.**	b	**40.**	c	**46.**	b	**52.**	c	**58.**	b
5.	b	**11.**	a	**17.**	c	**23.**	a	**29.**	d	**35.**	c	**41.**	b	**47.**	b	**53.**	c	**59.**	a
6.	a	**12.**	b	**18.**	d	**24.**	d	**30.**	a	**36.**	b	**42.**	d	**48.**	c	**54.**	a	**60.**	a

Combinations – Multiple-choice Solutions

1. $6! = 720$ Answer is c.

2. $\dfrac{1000!}{998!} = \dfrac{1000 \cdot 999 \cdot 998!}{998!} = 1000 \cdot 999 = 999000$ Answer is c.

3. The sum of coefficients in $(x+y)^n$ is $2^n = 2^5 = 32$. Answer is b.

4. The sum of numbers is $2^{n-1} = 2^{19} = 524288$. Answer is b.

5. One less than the number of students, $5! = 120$. Answer is b.

6. By the counting principle, $\dfrac{8!}{3!\,2!} = 3360$. Answer is a.

7. ${}_nC_r = {}_nC_{n-r}$ Answer is d.

8. ${}_8C_0\left(x^2\right)^8\left(-\dfrac{1}{y}\right)^0 + {}_8C_1\left(x^2\right)^7\left(-\dfrac{1}{y}\right)^1 + {}_8C_2\left(x^2\right)^6\left(-\dfrac{1}{y}\right)^2 = x^{16} - \dfrac{8x^{14}}{y} + \dfrac{28x^{12}}{y^2}$ Answer is c.

9. ${}_7C_3 \times {}_5C_2 = 350$ Answer is d.

10. ${}_5C_3 \times {}_6C_2 = 150$ Answer is d.

11. $9 \cdot 10 \cdot 10 \cdot 10 \cdot 25 \cdot 25 = 5625000$ Answer is a.

12. A to B to C to B to $A = 3 \cdot 3 \cdot 2 \cdot 2 = 36$ Answer is b.

13. To end up with 5 loonies when starting with 3 loonies, you must win 3 times and lose 1 time, therefore, $\dfrac{4!}{3!\,1!} = 4$. Answer is d.

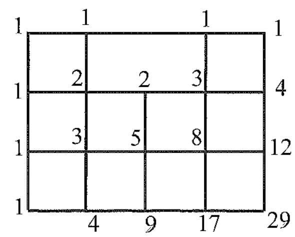
14. ${}_5C_3 \times {}_5C_2 = 10 \times 10 = 100$ Answer is c.

15. By the counting principle, $\dfrac{7!}{3!\,4!} \times \dfrac{4!}{2!\,2!} = 210$. Answer is a.

16. By the permutation with repetition rule, $\dfrac{15!}{6!\,5!\,4!} = 630630$. Answer is c.

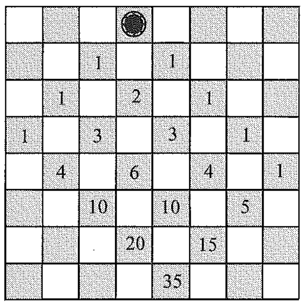
17. $P(\text{4 hearts and 1 non-heart}) = {}_{13}C_4 \times {}_{39}C_1 = 27885$. Answer is c.

18. The term containing x^5y^2 implies $t_3 = {}_7C_2(3x)^5(-2y)^2 = 20412$. Answer is d.

19. 

Answer is c.

20. By the counting principle: $\left(\text{A to B, } \dfrac{6!}{3!\,3!}\right) \times \left(\text{B to C, } \dfrac{3!}{2!\,1!}\right) = 60$. Answer is c.

21. 

Answer is a.

22. By factorial notation, $4! \times 3! = 144$. Answer is b.

23. By the combination principle, ${}_4C_4 \times {}_6C_2 = 15$. Answer is a.

24. $\dfrac{(n+2)!}{n! + 2(n-1)!} = \dfrac{(n+2)(n+1)(n)(n-1)!}{n(n-1)! + 2(n-1)!} = \dfrac{(n+2)(n+1)(n)(n-1)!}{(n-1)!\,(n+2)} = n(n+1) = n^2 + n$ Answer is d.

25. ${}_nP_2 = 14520 \rightarrow \dfrac{n!}{(n-2)!} = 14520 \rightarrow \dfrac{n(n-1)(n-2)!}{(n-2)!} = 14520 \rightarrow n(n-1) = 14520$, could use reasoning or $n^2 - n - 14520 = 0 \rightarrow (n-121)(n+120) = 0$, $n = 121$. Answer is a.

26. 2 women, 1 man or 3 women

${}_9C_2 \times {}_6C_1 + {}_9C_3 = 300$ Answer is d.

27. 10 letters: 3 "Ss", 2 "As", 2 "Ws", 3 single letters: $\dfrac{10!}{3!\ 2!\ 2!} = 151\,200$ Answer is b.

28. If Linda is one of the teachers selected, then ${}_7C_2 = 21$ other choices. Answer is a.

29. The first term is n, the second term is $n-1$, the third term is $n-2$, therefore, the r^{th} term is $n-(r-1) = n-r+1$. Answer is d.

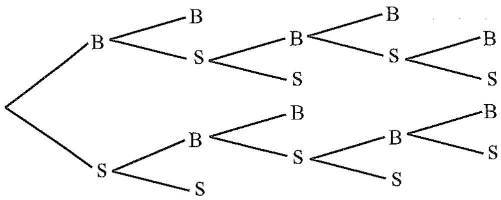
30. $(2a-3b^2)^8$ has 9 terms so the middle term is the 5^{th} term.

$t_{k+1} = {}_nC_k x^{n-k} \cdot y^k = t_5 = {}_8C_4(2a)^4(-3b^2)^4$ Answer is a.

31. $\dfrac{(n-1)!}{(n-3)!} - \dfrac{n!}{(n-2)!} = \dfrac{(n-1)(n-2)(n-3)!}{(n-3)!} - \dfrac{n(n-1)(n-2)!}{(n-2)!} = (n-1)(n-2) - n(n-1) =$

$n^2 - 3n + 2 - n^2 + n = -2n + 2$ Answer is c.

32. There are two choices for each coin, use the coin in your sum of money or don't use the coin. So the number of subsets is $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 - 1 = 31$ (you subtract one because you have to have at least one coin), answer is a.

33. 

Count the ends of the branches.
10 different outcomes are possible.
Answer is d.

34. This binomial has 7 terms, so we want the 6[th] term.

$t_{k+1} = {}_nC_k x^{n-k} \cdot y^k$ so $t_6 = {}_6C_5 \cdot x^1 \cdot (-2y)^5 = -192xy^5$ Answer is b.

35. If the binomial is to the 9[th] power, the variables (x^2) and $\left(\frac{1}{x}\right)$ have powers $(9-k)$ and k.

By inspection, $\left(x^2\right)^3 \times \left(\frac{1}{x}\right)^6$ gives a constant. Therefore $t_7 = {}_9C_6 \cdot \left(x^2\right)^3 \left(\frac{1}{x}\right)^6 = 84$. Answer is c.

36. ${}_nP_3 = 20n = \dfrac{n!}{(n-3)!} \rightarrow \dfrac{n(n-1)(n-2)(n-3)!}{n-3!} = 20n \rightarrow n(n-1)(n-2) = 20n \rightarrow$ reject solution $n = 0$

$n^2 - 3n + 2 = 20 \rightarrow n^2 - 3n - 18 = 0 \rightarrow (n-6)(n+3) = 0$, $n = -3, 0, 6$ reject $-3, 0$. Answer is b.

37. 3 white or 3 black $= {}_7C_3 + {}_5C_3 = 45$. Answer is b.

38. $\dfrac{n! - 2(n-2)!}{(n-2)(n-2)!} = \dfrac{n(n-1)(n-2)! - 2(n-2)!}{(n-2)(n-2)!} = \dfrac{n(n-1)-2}{n-2} = \dfrac{n^2-n-2}{n-2} = \dfrac{(n-2)(n+1)}{n-2} = n+1$.

Answer is b.

39. The pair of aces can be selected ${}_4C_2$ ways; the pair can be selected ${}_4C_2 \times 12$ ways; and the single card ${}_4C_1 \times 11$ ways, therefore ${}_4C_2 \times {}_4C_2 \times {}_4C_1 \times 12 \times 11 = 19008$ ways. Answer is d.

40. Three different pairs of shoes are ${}_{20}C_3 = 1140$. Answer is c.

41. All the pairs of shoes *minus* no black shoes = at least one black pair, therefore ${}_{20}C_3 - {}_8C_0 \times {}_{12}C_3 = 920$ ways or $P(1\text{ black}) + P(2\text{ black}) + P(3\text{ black}) = {}_8C_1 \times {}_{12}C_2 + {}_8C_2 \times {}_{12}C_1 + {}_8C_3 \times {}_{12}C_0 = 920$ ways. Answer is b.

42. At most 1 black pair of shoes means 0 or 1 black pair ${}_8C_0 \times {}_{12}C_3 + {}_8C_1 \times {}_{12}C_2 = 748$ ways. Answer is d.

43. At most 1 club means 0 or 1 club, therefore ${}_{13}C_0 \times {}_{39}C_5 + {}_{13}C_1 \times {}_{39}C_4 = 1\,645\,020$ ways. Answer is d.

44. The seven digits can be arranged in $7! = 5040$ ways. The digit 1 is before the digit 7 half the time so

$5040 \div 2 = 2520$. Answer is b.

45. By the fundamental counting principle $4! \times 3! \times 2! = 288$, but 3 items can be arranged in $3! = 6$ ways with half of these 6 ways having math books to the left of physics book, therefore $288 \times 3 = 864$. Answer is d.

46. $t_{k+1} = {}_nC_k x^{n-k} \cdot y^k$. If the 10[th] term of $\left(x^2-1\right)^n$ is $-55x^4$ then $t_{10} = \left({}_nC_9\right)\left(x^2\right)^{n-9}(-1)^9 = -55x^4$ equating exponents $2(n-9) = 4 \rightarrow n-9 = 2 \rightarrow n = 11$. Answer is b.

47. Andy's first two choices are a combination ${}_{10}C_2 = 45$. Answer is b.

48. Jerry has 8 choices, then 6 choices, then 4 choices = $8\times6\times4 = 192$. Jerry has no choice for his last two games. Answer is c.

49. $\dfrac{3n(n-2)!}{(n-3)!} = 105 \rightarrow \dfrac{3n(n-2)(n-3)!}{(n-3)!} = 105 \rightarrow 3n(n-2) = 105 \rightarrow 3n^2 - 6n - 105 = 0 \rightarrow$

$n^2 - 2n - 35 = 0 \rightarrow (n-7)(n+5) = 0$, $n = 7$, -5, reject -5. Answer is a.

50. Six students can arrange themselves in $5! = 120$ ways around a circular table. Two students could sit together in $4! = 24$ ways $\times\ 2 = 48$ since AB and BA are different. Therefore, not sitting together is $5! - 2 \times 4! = 72$. Answer is b.

51. The three boys and their dates can sit $3!$ ways with each boy and girl sitting $2!$ ways, therefore, $3! \times (2!)^3 = 48$. Answer is d.

52. The boys and girls can each sit $3!$ ways, therefore, $3! \times 3! = 36$, but we can sit boy then girl or girl then boy, so $36 \times 2 = 72$. Answer is c.

53. A man must be in the first seat, therefore, $4! = 24$ ways for men and $3! = 6$ ways for women = $24 \times 6 = 144$. Answer is c.

54. There are 20 first choice, 18 second choice and 16 third choice, but order does not make a difference, therefore, divide by 3!, $\dfrac{20\times18\times16}{3!} = 960$. Answer is a.

55. $\binom{6}{6} + \binom{6}{5} + \binom{6}{4} + \binom{6}{3} + \binom{6}{2} + \binom{6}{1} = 1+6+15+20+15+6 = 63$ Answer is c.

56. Consider the three forwards a one player, then the four players can line up 4! ways. Also, the three forwards can line up 3! ways. Therefore $4! \times 3! = 144$ ways. Answer is c.

57. The team must consist of 3 boys and 4 girls, or 4 boys and 3 girls.

Therefore ${}_{10}C_3 \times {}_8C_4 + {}_{10}C_4 \times {}_8C_3 = 20160$ ways. Answer is b.

58. $\dfrac{{}_nP_r}{{}_nC_r} = \dfrac{\dfrac{n!}{(n-r)!}}{\dfrac{n!}{r!(n-r)!}} = r!$, therefore, $\dfrac{55440}{462} = r! \rightarrow r! = 120 \rightarrow r = 5$ Answer is b

59. In your total, you may use 0, 1, or 2 pennies, 0, 1, or 2 dimes and 0, 1, 2, 3 quarters but you must have at least one coin, therefore, $3 \times 3 \times 4 - 1 = 35$. Answer is a.

60. ${}_{n-1}C_{r-1} + {}_{n-1}C_r = \dfrac{(n-1)!}{(r-1)!\,((n-1)-(r-1))!} + \dfrac{(n-1)!}{r!(n-1-r)!} = \dfrac{(n-1)!}{(r-1)!\,(n-r)!} + \dfrac{(n-1)!}{r!(n-r-1)!} =$

$\dfrac{(n-1)!}{(r-1)!\,(n-r)(n-r-1)!} + \dfrac{(n-1)!}{r(r-1)!\,(n-r-1)!} = \dfrac{r(n-1)! + (n-r)(n-1)!}{r(n-r)(r-1)!(n-r-1)!} =$

$\dfrac{n(n-1)!}{r(n-r)(r-1)!(n-r-1)!} = \dfrac{n!}{r!(n-r)!} = {}_nC_r$ Answer is a.