# Ch 7 – Trig Identities

## 7.1 Reciprocal and Quotient Identities

### Quotient Identities

$$\tan x = \frac{\sin x}{\cos x} \qquad\qquad \cot x = \frac{\cos x}{\sin x}$$

Ex. Solve $\sin x = \cos x$ for $0 \le x < 2\pi$.

For this question, we will use the quotient identity, $\tan x = \frac{\sin x}{\cos x}$

$\frac{\sin x}{\cos x} = \frac{\cos x}{\cos x}$ divide both side by $\cos x$

$\tan x = 1$ $\tan x > 0$, so solutions in QI and QIII

$x_1 = \frac{\pi}{4}$ solution is in QI, and is also $\theta_r$

$x_2 = \pi + \frac{\pi}{4} = \frac{5\pi}{4}$ solution in QIII

$\therefore x = \frac{\pi}{4}, \frac{5\pi}{4}$

### Reciprocal Identities

$$\sin x = \frac{1}{\csc x} \qquad \cos x = \frac{1}{\sec x} \qquad \tan x = \frac{1}{\cot x}$$

$$\csc x = \frac{1}{\sin x} \qquad \sec x = \frac{1}{\cos x} \qquad \cot x = \frac{1}{\tan x}$$

Using a combination of reciprocal and quotient identities we can get:

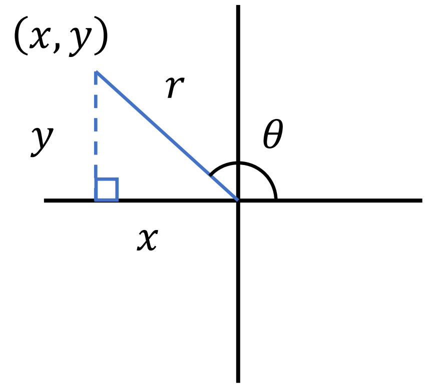
$$\tan x = \frac{\sec x}{\csc x} \qquad\qquad \cot x = \frac{\csc x}{\sec x}$$

## Proof of a Pythagorean Identity

For an angle $\theta$ in standard position with terminal arm of length $r$, and coordinate at $(x, y)$:

Begin with Pythagorean Theorem

$$x^2 + y^2 = r^2$$




Divide both sides by $r^2$

$$\frac{x^2}{r^2} + \frac{y^2}{r^2} = \frac{r^2}{r^2}$$

$$\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = 1$$

Since $\sin\theta = \frac{y}{r}$ and $\cos\theta = \frac{x}{r}$, after some rearranging and substitutions, we get:

$$\sin^2 x + \cos^2 x = 1$$

## Pythagorean Identities

$$\sin^2 x + \cos^2 x = 1 \qquad 1 + \tan^2 x = \sec^2 x$$

$$1 + \cot^2 x = \csc^2 x$$

## Techniques for Trig Proofs

For this chapter, the following are helpful techniques to simplify / solve a trigonometric expression or equation.

1. **Common Denominator**

$$\frac{2}{x} + 3$$

$$= \frac{2}{x} + \frac{3x}{x}$$

$$= \frac{2+3x}{x}$$

$$\frac{1}{\sin x} + \cos x$$

$$= \frac{1}{\sin x} + \frac{\sin x \cos x}{\sin x}$$

$$= \frac{1+\sin x \cos x}{\sin x}$$

2. **Factoring**

$x^2 - 9$

$= (x+3)(x-3)$

$\cos^2 x - 1$

$= (\cos x + 1)(\cos x - 1)$

$2x^2 + 7x + 3$

$= (2x+1)(x+3)$

$2\sin^2 x + 7\sin x + 3$

$= (2\sin x + 1)(\sin x + 3)$

3. **Change into Sine and Cosine**

$\frac{\tan x}{\sec x}$

$= \frac{\frac{\sin x}{\cos x}}{\frac{1}{\cos x}}$

$= \sin x$

$\frac{\sin x}{\csc x} + \frac{\cos x}{\sec x}$

$= \frac{\sin x}{\frac{1}{\sin x}} + \frac{\cos x}{\frac{1}{\cos x}}$

$= \sin^2 x + \cos^2 x$

$= 1$

4. **Conjugate Pairs**

$\frac{4}{\sqrt{3}-1}$

$= \frac{4}{\sqrt{3}-1} \times \frac{\sqrt{3}+1}{\sqrt{3}+1}$

$= \frac{4(\sqrt{3}+1)}{3-1}$

$= \frac{4(\sqrt{3}+1)}{2}$

$= 2(\sqrt{3}+1)$ or $2\sqrt{3}+2$

$\frac{1}{1-\sin x}$

$= \frac{1}{1-\sin x} \times \frac{1+\sin x}{1+\sin x}$

$= \frac{1+\sin x}{1-\sin^2 x}$

$= \frac{1+\sin x}{\cos^2 x}$

Recall: $\cos^2 x = 1 - \sin^2 x$

Ex. Simplify $\frac{1+\sin x}{\cos x} - \frac{\cos x}{1-\sin x}$ Write as a single fraction.

Multiply by the conjugate, and anticipate that $1 - \sin^2 x = \cos^2 x$

$$= \frac{(1+\sin x)(\cos x)}{(\cos x)(\cos x)} - \frac{\cos x(1+\sin x)}{(1-\sin x)(1+\sin x)}$$

$$= \frac{\cos x+\sin x \cos x}{\cos^2 x} - \frac{\cos x+\sin x \cos x}{1-\sin^2 x}$$

Use Pythagorean Identity $\sin^2 x + \cos^2 x = 1$

$$= \frac{\cos x+\sin x \cos x}{\cos^2 x} - \frac{\cos x+\sin x \cos x}{\cos^2 x}$$

$$= \frac{\cos x+\sin x \cos x-\cos x-\sin x \cos x}{\cos^2 x}$$

$$= 0$$

Alternatively, use common denominator

$$= \frac{1+\sin x}{\cos x} \cdot \frac{1-\sin x}{1-\sin x} - \frac{\cos x}{1-\sin x} \cdot \frac{\cos x}{\cos x}$$

$$= \frac{1-\sin^2 x}{\cos x(1-\sin x)} - \frac{\cos^2 x}{(1-\sin x)\cos x}$$

Factor out the negative sign from $-\sin^2 x - \cos^2 x$

$$= \frac{1-\sin^2 x-\cos^2 x}{\cos x(1-\sin x)}$$

Use Pythagorean Identity $\sin^2 x + \cos^2 x = 1$

$$= \frac{1-(\sin^2 x+\cos^2 x)}{\cos x(1-\sin x)}$$

$$= \frac{1-1}{\cos x(1-\sin x)}$$

$$= 0$$

Ex. Determine the restriction on $\tan x + \csc x$, for $0 \leq x < 2\pi$

Restrictions are necessary when there is a fraction
Change into sine and cosine

$$= \tan x + \csc x$$

$$= \frac{\sin x}{\cos x} + \frac{1}{\sin x}$$

$$= \frac{\sin^2 x + \cos x}{\sin x \cos x}$$

We need to ensure that $\cos x \neq 0$ and $\sin x \neq 0$

For $\cos x \neq 0 \ \rightarrow x \neq \frac{\pi}{2}, \frac{3\pi}{2}$ $\qquad \sin x \neq 0 \ \rightarrow x \neq 0, \pi$

Restrictions: $x \neq 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$

Ex. Simplify $\dfrac{\frac{\cot x + 1}{\cot x} - 1}{\frac{\cot x - 1}{\cot x} - 1}$

Use common denominator, and then simplify the complex fraction

$$= \frac{\frac{\cot x + 1}{\cot x} - \frac{\cot x}{\cot x}}{\frac{\cot x - 1}{\cot x} - \frac{\cot x}{\cot x}}$$

$$= \frac{\frac{\cot x + 1 - \cot x}{\cot x}}{\frac{\cot x - 1 - \cot x}{\cot x}}$$

$$= \frac{\frac{1}{\cot x}}{\frac{-1}{\cot x}}$$

$$= -1$$

Ex. Simplify $\frac{1-\sec^2 x}{\sec^2 x} - \cos^2 x$

Get common denominator by using reciprocal identity

$$= \frac{1-\sec^2 x}{\sec^2 x} - \frac{1}{\sec^2 x}$$

$$= \frac{-\sec^2 x}{\sec^2 x}$$

$$= -1$$

Alternatively, change into sine and cosine, and use common denominator.

$$= \frac{1-\frac{1}{\cos^2 x}}{\frac{1}{\cos^2 x}} - \cos^2 x$$

$$= \frac{\frac{\cos^2 x}{\cos^2 x}-\frac{1}{\cos^2 x}}{\frac{1}{\cos^2 x}} - \cos^2 x$$

Simplify the complex fraction

$$= \frac{\frac{\cos^2 x-1}{\cos^2 x}}{\frac{1}{\cos^2 x}} - \cos^2 x$$

$$= \cos^2 x - 1 - \cos^2 x$$

$$= -1$$

**7.1 Homework:**

#2-9 bcf…

## 7.2 Verifying Trigonometric Identities

### Proving Trig Identities

Goal: Prove that the left side is equal to the right side

Strategy: Using Trig identities, we modify one or both sides so that they show the same expression
Decide which side to begin (choose the more complicated side)

Note: Unlike solving an equation, you cannot apply +,-, x, ÷ to both sides
It is not permitted to move a term or expression, from one side of the proof to the other side.

Ex. Prove $\frac{\csc^2 x - 1}{\csc^2 x} = \cos^2 x$

| $\frac{\csc^2 x - 1}{\csc^2 x}$ | $\cos^2 x$ |
|---|---|
| $\frac{\cot^2 x}{\csc^2 x}$ | |
| $\frac{\frac{\cos^2 x}{\sin^2 x}}{\frac{1}{\sin^2 x}}$ | |
| $\frac{\cos^2 x}{\sin^2 x} \times \frac{\sin^2 x}{1}$ | |
| $\cos^2 x$ | |

LS = RS

It is not necessary to simplify only one side of the proof, although it looks better!

Ex. Prove $\frac{1}{1-\cos x}+\frac{1}{1+\cos x}=2\csc^2 x$

| $\frac{1}{1-\cos x}+\frac{1}{1+\cos x}$ | $2\csc^2 x$ |
| --- | --- |
| $\frac{1+\cos x}{(1-\cos x)(1+\cos x)}+\frac{1-\cos x}{(1-\cos x)(1+\cos x)}$ | |
| $\frac{2}{1-\cos^2 x}$ | |
| $\frac{2}{\sin^2 x}$ | |
| $2\csc^2 x$ | |

LS = RS

Ex. Prove $\tan x+\cot x=\sec x\csc x$

| $\tan x+\cot x$ | $\sec x\csc x$ |
| --- | --- |
| $\frac{\sin x}{\cos x}+\frac{\cos x}{\sin x}$ | |
| $\frac{\sin^2 x}{\sin x\cos x}+\frac{\cos^2 x}{\sin x\cos x}$ | |
| $\frac{\sin^2 x+\cos^2 x}{\sin x\cos x}$ | |
| $\frac{1}{\sin x\cos x}$ | |
| $\sec x\csc x$ | |

LS = RS

Ex. Prove $\frac{\tan^2 x}{1+\sec x} = \frac{1-\cos x}{\cos x}$

| $\frac{\tan^2 x}{1+\sec x}$ | $\frac{1-\cos x}{\cos x}$ |
|---|---|
| $\frac{\sec^2 x-1}{1+\sec x}$ | |
| $\frac{(\sec x+1)(\sec x-1)}{1+\sec x}$ | |
| $\sec x - 1$ | |
| $\frac{1}{\cos x} - \frac{\cos x}{\cos x}$ | |
| $\frac{1-\cos x}{\cos x}$ | |

LS = RS

Ex. Prove $\frac{\cos^4 x-\sin^4 x}{1-\tan^4 x} = \cos^4 x$

| $\frac{\cos^4 x-\sin^4 x}{1-\tan^4 x}$ | $\cos^4 x$ |
|---|---|
| $\frac{\cos^4 x-\sin^4 x}{1-\frac{\sin^4 x}{\cos^4 x}}$ | |
| $\frac{\cos^4 x-\sin^4 x}{\frac{\cos^4 x}{\cos^4 x}-\frac{\sin^4 x}{\cos^4 x}}$ | |
| $\frac{\cos^4 x-\sin^4 x}{\frac{\cos^4 x-\sin^4 x}{\cos^4 x}}$ | |
| $\frac{(\cos^4 x-\sin^4 x)(\cos^4 x)}{\cos^4 x-\sin^4 x}$ | |
| $\cos^4 x$ | |

LS = RS

Ex. Prove $\frac{\sec^4 x-1}{\tan^2 x} = 2+\tan^2 x$

| $\frac{\sec^4 x-1}{\tan^2 x}$ | $2+\tan^2 x$ |
|---|---|
| $\frac{(\sec^2 x+1)(\sec^2 x-1)}{\sec^2 x-1}$ | |
| $\sec^2 x+1$ | |
| $\tan^2 x+1+1$ | |
| $\tan^2 x+2$ | |

LS = RS

Ex. Prove $\csc^2\left(\frac{\pi}{2}-x\right)-1=\tan^2 x$

| $\csc^2\left(\frac{\pi}{2}-x\right)-1$ | $\tan^2 x$ |
|---|---|
| $\frac{1}{\sin^2\left(\frac{\pi}{2}-x\right)}-1$ | $\sec^2 x-1$ |
| Note: $\sin\left(\frac{\pi}{2}-x\right)=\cos x$ | |
| $\cos\left(\frac{\pi}{2}-x\right)=\sin x$ | |
| $\frac{1}{\cos^2 x}-1$ | |
| $\sec^2 x-1$ | |

LS = RS

Ex. Prove $\frac{2\sin x\cos x+1}{\sin^2 x-\cos^2 x}=\frac{1+\cot x}{1-\cot x}$

| $\frac{2\sin x\cos x+1}{\sin^2 x-\cos^2 x}$ | $\frac{1+\cot x}{1-\cot x}$ |
|---|---|
| $\frac{2\sin x\cos x+1}{(\sin x+\cos x)(\sin x-\cos x)}$ | |
| $\frac{2\sin x\cos x+\sin^2 x+\cos^2 x}{(\sin x+\cos x)(\sin x-\cos x)}$ | $\leftarrow 1=\sin^2 x+\cos^2 x$ |
| $\frac{\sin^2 x+2\sin x\cos x+\cos^2 x}{(\sin x+\cos x)(\sin x-\cos x)}$ | |
| $\frac{(\sin x+\cos x)(\sin x+\cos x)}{(\sin x+\cos x)(\sin x-\cos x)}$ | |
| $\frac{\sin x+\cos x}{\sin x-\cos x}$ | |
| $\frac{1+\frac{\cos x}{\sin x}}{1-\frac{\cos x}{\sin x}}$ | |
| $\frac{1+\cot x}{1-\cot x}$ | |

LS = RS

**7.2 Homework:**

# 2, 3, 6, 7, 10, 11, 14, 16, 17, 20, 22, 23, 26

## 7.3 Trigonometric Equations

### Solving Trigonometric Equations and General Solutions

We have already solved trigonometric equations where the solutions were restricted to be between zero and $2\pi$.

When solving for general solutions, there is no restriction for the domain; need to solve for all possible solutions.

Ex. Solve $2\cos x + 1 = 0$

a. For $0 \leq x < 2\pi$

b. General Solutions

a. For $0 \leq x < 2\pi$

We use the same technique shown in the previous chapter.

It is possible to use substitution to make the equation a bit easier to solve. Use the variable $y$ to represent $\cos x$.

Let $y = \cos x$

$$2y + 1 = 0$$

$$y = -\frac{1}{2}$$

Re-substitute $\cos x$ for $y$, and solve for $x$.

$$\cos x = -\frac{1}{2}$$

There are two angles that make the cosine ratio equal to $-\frac{1}{2}$.
These solutions are in QII and QIII. The reference angle is $\frac{\pi}{3}$.

$x_1 = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$ solution in QII

$x_2 = \pi + \frac{\pi}{3} = \frac{4\pi}{3}$ solution in QIII

$$\therefore x = \frac{2\pi}{3}, \frac{4\pi}{3}$$

b. General Solution

To determine the general solution, we need to find all coterminal angles for each solution in part a.

First, find coterminal angles for $x_1$. To find coterminal angles, add or subtract $2\pi$.

In the counterclockwise direction:

$$x_1 = \frac{2\pi}{3} + 2\pi, \frac{2\pi}{3} + 4\pi, \frac{2\pi}{3} + 6\pi, \ldots$$

In the clockwise direction:

$$x_1 = \frac{2\pi}{3} - 2\pi, \frac{2\pi}{3} - 4\pi, \frac{2\pi}{3} - 6\pi, \ldots$$

Combing the original solution, and answers in both counterclockwise and clockwise direction:

$$x_1 = \cdots, \frac{2\pi}{3} - 4\pi, \frac{2\pi}{3} - 2\pi, \frac{2\pi}{3}, \frac{2\pi}{3} + 2\pi, \frac{2\pi}{3} + 4\pi, \frac{2\pi}{3} + 6\pi, \ldots$$

Re-writing the solution with $2\pi$ factored, a pattern emerges.

$$x_1 = \cdots, \frac{2\pi}{3} + (-1)2\pi, \frac{2\pi}{3} + (0)2\pi, \frac{2\pi}{3} + (1)2\pi, \frac{2\pi}{3} + (2)2\pi, \ldots$$

To generalize this pattern, we can summarize all the numbers in the parentheses with $n$, where $n$ is an integer.

$$x_1 = \frac{2\pi}{3} + 2\pi n, n \in \mathbb{Z}$$

Using the same logic, the general solution for $x_2$:

$$x_2 = \frac{4\pi}{3} + 2\pi n, n \in \mathbb{Z}$$

Combining the two solutions:

$$x = \frac{2\pi}{3} + 2\pi n, \frac{4\pi}{3} + 2\pi n, n \in \mathbb{Z}$$

**Solving Trigonometric Equations with Trigonometric Identities**

The following question requires the use of trigonometric identities in order to solve the equation.

Ex. Solve $2\cos^2 x + 3\sin x - 3 = 0$

a. For $0 \leq x < 2\pi$

b. For the general solution

a. For $0 \leq x < 2\pi$

If possible, find a way to have a single trig function in the equation.

Use Pythagorean Identity and substitute $\cos^2 x$ for $1 - \sin^2 x$

$$2(1 - \sin^2 x) + 3\sin x - 3 = 0$$

Next, simplify and factor.

$$2 - 2\sin^2 x + 3\sin x - 3 = 0$$

$$2\sin^2 x - 3\sin x + 1 = 0$$

$$(2\sin x - 1)(\sin x - 1) = 0$$

$$\sin x = \frac{1}{2}, 1$$

Next, solve for $x$.

$$\sin x = \frac{1}{2}$$

$x$ is in QI and QII

$$x = \frac{\pi}{6}, \frac{5\pi}{6}$$

$$\sin x = 1$$

Use sine wave

$$x = \frac{\pi}{2}$$

Combine the solutions.

$$\therefore x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$$

b. General Solution

Write a general solution for each of the solutions in part a.

$$x = \frac{\pi}{6} + 2\pi n, \frac{\pi}{2} + 2\pi n, \frac{5\pi}{6} + 2\pi n, n \in \mathbb{Z}$$

**Solving Trig Equations with Horizontal Compressions (or Expansions)**

The horizontal compressions/expansions with affect the number of possible solutions in the restricted domain from $0$ to $2\pi$.

Ex. Solve $6\sin^2 2x - \sin 2x - 1 = 0$

a. For $0 \le x < 2\pi$

b. general solutions

a. For $0 \le x < 2\pi$

Do not use the double angle identity, instead treat $2x$ as if it was $\theta$. The $2x$ will be dealt with later. Start by factoring.

Let $\theta = 2x$

$$(3\sin\theta + 1)(2\sin\theta - 1) = 0$$

$$3\sin\theta + 1 = 0$$

$$\sin\theta = -\frac{1}{3}$$

The solutions are in QIII and QIV

$$\theta = \sin^{-1}\left(-\frac{1}{3}\right)$$

$$\theta_2 = -0.3398$$

This solution is in QIV, add $2\pi$ to get positive coterminal solution

$$\theta_2 = -0.3398 + 2\pi$$

$$\theta_2 = 5.9433$$

$$\theta_r = 2\pi - 5.9433$$

Find the reference in order to solve for the solution in QIII

$$\theta_r = 0.3398$$

$$\theta_1 = \pi + \theta_r$$

$$\theta_1 = 3.4814$$

This is the solution in QIII

$$\theta = 3.4814, 5.9433$$

Now, we need to deal with the "$2x$".

In terms of transformations, $y = \sin 2x$, experiences a horizontal compression by a factor of $\frac{1}{2}$. Normally, one period of sine wave fits into the domain $0$ to $2\pi$. A horizontal compression by a factor of $\frac{1}{2}$ allows for two periods of the sine wave to fit into the previous domain.

So, while $\sin\theta = -\frac{1}{3}$ has 2 solutions where $0 \le \theta < 2\pi$, $\sin 2x = -\frac{1}{3}$ is expected to have 2 sets of solutions, 4 in total.

To find the additional set of solutions, just find the coterminal angles for the previous two solutions.

$$3.4814 + 2\pi = 9.7646 \qquad\qquad 5.9433 + 2\pi = 12.2265$$

There are now 4 solutions for $2x$. Divide everything by 2 to solve for $x$.

$$2x = 3.4814, 5.9433, 9.7646, 12.2265$$
$$x = 1.741, 2.972, 4.883, 6.113$$

Next, solve the other part of the trig equation in a similar manner.

$$2\sin 2x - 1 = 0$$

$$\sin 2x = \frac{1}{2}$$ Solutions are in QI and QII

$$2x = \sin^{-1}\left(\frac{1}{2}\right)$$

$$2x = \frac{\pi}{6}, \frac{5\pi}{6}$$ to get the second set of solutions, add $2\pi$

$$2x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}$$

$$x = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}$$

All of the solutions:

$$x = 1.741, 2.972, 4.883, 6.113, \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}$$

b. General Solution

$$2x = 3.4814 + 2\pi n, 5.9433 + 2\pi n, \frac{\pi}{6} + 2\pi n, \frac{5\pi}{6} + 2\pi n, n \in \mathbb{Z}$$

$$x = 1.741 + \pi n, 2.972 + \pi n, \frac{\pi}{12} + \pi n, \frac{5\pi}{12} + \pi n, n \in \mathbb{Z}$$

**The Number of Solutions for $\sin bx$ and $\cos bx$**

In general, when solving an equation involve $\sin bx$ or $\cos bx$, the number of solutions is $2b$ when $b > 1$ and $b \in \mathbb{W}$ is.

For a rational number $\frac{a}{b}$, where $a < b$:

$\sin 2x = \frac{a}{b}$ should have 4 solutions

$\sin 3x = \frac{a}{b}$ should have 6 solutions

$\sin 4x = \frac{a}{b}$ should have 8 solutions

Ex. Solve $\cos 3x = \frac{\sqrt{2}}{2}$ for $0 \leq x < 2\pi$.

$3x = \cos^{-1}\left(\frac{\sqrt{2}}{2}\right)$ Since $b = 3$, expect to have 6 solutions

$3x = \frac{\pi}{4}, \frac{7\pi}{4}$ Solutions in QI and QIV

Find the additional solutions by adding $2\pi$

$$3x = \frac{\pi}{4}, \frac{7\pi}{4}, \frac{9\pi}{4}, \frac{15\pi}{4}, \frac{17\pi}{4}, \frac{23\pi}{4}$$

Divide everything by 3

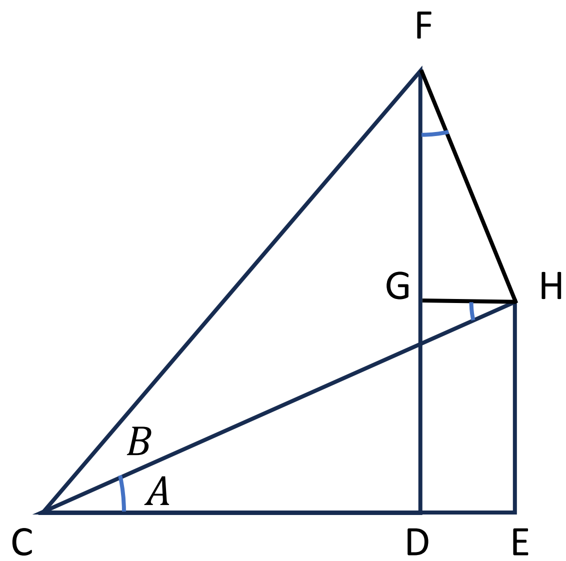
$$x = \frac{\pi}{12}, \frac{7\pi}{12}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{17\pi}{12}, \frac{23\pi}{12}$$

**7.3 Homework:**

# 1-5 bcf…, 6ac

## 7.4 Sum and Difference Identities

**Sum Identity of Sine:** $\boxed{\boldsymbol{\sin(A+B) = \sin A \cos B + \cos A \sin B}}$




From parallel line rules:

$$\angle A = \angle CHG$$

From angle sum of a triangle:

$$\angle CHG = \angle GFH$$

Proof:

$\sin(A+B)$

$= \frac{DF}{CF}$

$= \frac{FG+DG}{CF}$

$= \frac{FG}{CF} + \frac{DG}{CF}$

$= \frac{FG}{CF} + \frac{EH}{CF}$

$= \frac{FG}{CF} \cdot \frac{FH}{FH} + \frac{EH}{CF} \cdot \frac{CH}{CH}$

$= \frac{FG}{FH} \cdot \frac{FH}{CF} + \frac{EH}{CH} \cdot \frac{CH}{CF}$

$= \cos A \sin B + \sin A \cos B$

$= \sin A \cos B + \cos A \sin B$

**Sum Identities**

The following are Sum Identities involving 2 angles, $A$ and $B$.

$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

**Difference Identities**

These are the Difference Identities involving 2 angles, $A$ and $B$.

$$\sin(A-B) = \sin A \cos B - \cos A \sin B$$

$$\cos(A-B) = \cos A \cos B + \sin A \sin B$$

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

Ex. Determine exact value of $\sin 165°$

Use sum or difference identity for sine

> convert 165 as a sum or difference of two "special angles"

Special Angles: 0, 30, 45, 60, 90, 120, 135,150,180…

Possible choices include:

$165 = 30 + 135$ $\qquad\qquad$ $165 = 45 + 120$

$\sin 165 = \sin(30 + 135)$

$A = 30$ $\qquad\qquad$ $B = 135$

$= \sin 30 \cos 135 + \cos 30 \sin 135$

$= \left(\frac{1}{2}\right)\left(-\frac{1}{\sqrt{2}}\right) + \left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{2}}\right)$

$= -\frac{1}{2\sqrt{2}} + \frac{\sqrt{3}}{2\sqrt{2}}$

$= \frac{\sqrt{3}-1}{2\sqrt{2}}$

Ex. Simplify $\dfrac{\tan\frac{2\pi}{5}-\tan\frac{3\pi}{20}}{1+\tan\frac{2\pi}{5}\tan\frac{3\pi}{20}}$

Let A = $\frac{2\pi}{5}$ and $B = \frac{3\pi}{20}$ Use: $\tan(A-B) = \frac{\tan A-\tan B}{1+\tan A\tan B}$

$= \tan\left(\frac{2\pi}{5}-\frac{3\pi}{20}\right)$

$= \tan\left(\frac{8\pi}{20}-\frac{3\pi}{20}\right)$

$= \tan\frac{5\pi}{20}$

$= \tan\frac{\pi}{4}$ same as $\tan 45°$

$= 1$

Ex. Given $\sin A = \frac{5}{13}$, $A$ is in II, and $\cos B = \frac{2}{3}$, $B$ is in IV, determine the exact value of $\cos(A-B)$.

$\cos(A-B) = \cos A \cos B + \sin A \sin B$

Need to find value of $\cos A$ and $\sin B$

A: $x^2 = 13^2 - 5^2$
$x = -12$
$\therefore \cos A = -\frac{12}{13}$

B: $y^2 = 3^2 - 2^2$
$y = -\sqrt{5}$
$\therefore \sin B = -\frac{\sqrt{5}}{3}$

$= \left(-\frac{12}{13}\right)\left(\frac{2}{3}\right) + \left(\frac{5}{13}\right)\left(-\frac{\sqrt{5}}{3}\right)$

$= \frac{-24}{39} + \frac{-5\sqrt{5}}{39}$

$= -\frac{24+5\sqrt{5}}{39}$

Ex. Solve $\sqrt{2}\sin 3x\cos 2x = 1 + \sqrt{2}\cos 3x\sin 2x$ for $0 \le x < 2\pi$

$$\sqrt{2}\sin 3x\cos 2x = 1 + \sqrt{2}\cos 3x\sin 2x$$

$$\sqrt{2}\sin 3x\cos 2x - \sqrt{2}\cos 3x\sin 2x = 1$$

$$\sin 3x\cos 2x - \cos 3x\sin 2x = \frac{1}{\sqrt{2}}$$

Let $3x$ be $A$ and $2x$ be $B$ in $\sin A\cos B - \cos A\sin B = \sin(A - B)$

$$\sin(3x - 2x) = \frac{1}{\sqrt{2}}$$

$\sin x = \frac{1}{\sqrt{2}}$ $\qquad$ $x$ is in QI and QII

Since the sine ratio is $\frac{1}{\sqrt{2}}$, this is the 45-45-90 triangle $\therefore x_r = \frac{\pi}{4}$

$x_1 = \frac{\pi}{4}$ $\qquad$ (QI)

$x_2 = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$ $\qquad$ (QII)

$$x = \frac{\pi}{4}, \frac{3\pi}{4}$$

Ex. Simplify $\tan^2\left(\frac{\pi}{2} - x\right)\sec^2 x - \sin^2\left(\frac{\pi}{2} - x\right)\csc^2 x$

$$= \cot^2 x\sec^2 x - \cos^2 x\csc^2 x$$

$$= \frac{\csc^2 x}{\sec^2 x} \times \sec^2 x - \frac{1}{\sec^2 x} \times \csc^2 x$$

$$= \csc^2 x - \cot^2 x$$

$$= 1$$

Ex. Prove $\sin(A+B) + \sin(A-B) = 2\sin A \cos B$

| | |
|---|---|
| $\sin(A+B) + \sin(A-B)$ | $2\sin A \cos B$ |
| $\sin A \cos B + \cos A \sin B + \sin A \cos B - \cos A \sin B$ | |
| $\sin A \cos B + \sin A \cos B$ | |
| $2\sin A \cos B$ | |

LS = RS

**7.4 Homework**

# 1-7 bcf…, 9

## 7.5 Double-Angle Identities

Ex. Show that $\sin 2A = 2\sin A\cos A$ using Sum Identity for Sine

Recall: $\sin(A+B) = \sin A\cos B + \cos A\sin B$

$$\sin 2A = \sin(A+A)$$

$$= \sin A\cos A + \cos A\sin A$$

$$= 2\sin A\cos A$$

**Double-Angle Identities**

| | |
|---|---|
| $\cos 2A = \cos^2 A - \sin^2 A$ | $\sin 2A = 2\sin A\cos A$ |
| $\cos 2A = 1 - 2\sin^2 A$ | $\tan 2A = \frac{2\tan A}{1-\tan^2 A}$ |
| $\cos 2A = 2\cos^2 A - 1$ | |

Ex. Prove the identity. $\frac{\sin 6x}{1+\cos 6x} = \tan 3x$

| | |
|---|---|
| $\frac{\sin 6x}{1+\cos 6x}$ | $\tan 3x$ |
| $> \sin 6x = \sin(2(3x))$ | |
| $\frac{2\sin 3x\cos 3x}{1+(2\cos^2 3x-1)}$ | $\frac{\sin 3x}{\cos 3x}$ |
| $\frac{2\sin 3x\cos 3x}{2\cos^2 3x}$ | |
| $\frac{\sin 3x}{\cos 3x}$ | |

LS = RS

Ex. Solve $\cos 2x = 2\sin^2 x, 0 \le x < 2\pi$

Note: in this scenario, try to reduce $2x \to x$
There are 3 choices for $\cos 2x$, choose $1 - 2\sin^2 x$

$1 - 2\sin^2 x = 2\sin^2 x$

$1 = 4\sin^2 x$

$\sin^2 x = \frac{1}{4}$

$\sin x = \pm\frac{1}{2}$

| | |
|---|---|
| $\sin x = \frac{1}{2}$ | $\sin x = -\frac{1}{2}$ |
| $x$ is in I, II | $x$ is in III, IV |
| $x = \frac{\pi}{6}, \pi - \frac{\pi}{6}$ | $x = \pi + \frac{\pi}{6}, 2\pi - \frac{\pi}{6}$ |

$\therefore x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$

Ex. Given $\sin\theta = -\frac{3}{5}$ in III, determine $\tan 2\theta$.

$\sin\theta = -\frac{3}{5} \quad \to \quad y = -3, \quad r = 5$

Using Pythagorean theorem $\therefore x = \pm 4$
Since $\theta$ is in III, $x = -4$

$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$ $\qquad$ $\tan\theta = \frac{-3}{-4} = \frac{3}{4}$

$$= \frac{2\left(\frac{3}{4}\right)}{1-\left(\frac{3}{4}\right)^2} = \frac{\frac{3}{2}}{\frac{7}{16}} = \frac{24}{7}$$

**Power-Reducing Identities**

$$\sin^2 A = \frac{1-\cos 2A}{2} \quad \text{or} \quad = \frac{1}{2}(1-\cos 2A)$$

$$\cos^2 A = \frac{1+\cos 2A}{2} \quad \text{or} \quad = \frac{1}{2}(1+\cos 2A)$$

$$\tan^2 A = \frac{1-\cos 2A}{1+\cos 2A}$$

Ex. Solve $\csc^2 x = 2\sec 2x$ for $0 \le x < 2\pi$

$$\csc^2 x = 2\sec 2x$$

$$\frac{1}{\sin^2 x} = \frac{2}{\cos 2x}$$

$$\cos 2x = 2\sin^2 x$$

$$1 - 2\sin^2 x = 2\sin^2 x$$

$$1 = 4\sin^2 x$$

$$\sin^2 x = \frac{1}{4}$$

$$\sin x = \pm\frac{1}{2}$$

$\sin x = \frac{1}{2}$ | $\sin x = -\frac{1}{2}$

$x$ is in I, II | $x$ is in III, IV

$x = \frac{\pi}{6}, \frac{5\pi}{6}$ | $x = \frac{7\pi}{6}, \frac{11\pi}{6}$

$$\therefore x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$$

Ex. Write $\cos 3x$ in terms of $\cos x$

$$= \cos(2x + x)$$

$$= \cos 2x \cos x - \sin 2x \sin x$$

$$= (2\cos^2 x - 1)\cos x - (2\sin x \cos x)\sin x$$

$$= 2\cos^3 x - \cos x - 2\sin^2 x \cos x$$

$$= 2\cos^3 x - \cos x - 2(1 - \cos^2 x)\cos x$$

$$= 2\cos^3 x - \cos x - 2\cos x + 2\cos^3 x$$

$$= 4\cos^3 x - 3\cos x$$

**7.5 Homework:**

Part I

# 1-2 bcf…, 3 bcf…, 4-5 ac, 7, 8, 10, 12, 15