Chapter 4 – Radicals and Rational Functions

4.1 Radicals

Review of Radicals

$x^n = a$ $x = \sqrt[n]{a}$	If a and x are real numbers and n is a positive integer, then x is an nth root of a if $x^n = a$

Index Number, n

when $n = 2$, square root $\sqrt{a}$

$n = 3$, cube root $\sqrt[3]{a}$

$n = 4$, 4th root $\sqrt[4]{a}$ and et cetera…

Radicand, a

The “inside” of a root/radical

When **n is even**, need to ensure $\boldsymbol{a \geq 0}$
If $a < 0$, the radical is an imaginary number

When **n is odd**, there are **no restrictions** on a

Basic Radical Facts

$\sqrt[n]{1} = 1$ $\qquad\qquad$ $\sqrt[n]{0} = 0$

$\sqrt[n]{x} = x^{\frac{1}{n}}$ $\qquad\qquad$ $\sqrt[n]{x^m} = x^{\frac{m}{n}}$

$\sqrt[n]{x^n} = x$ when n is odd $\qquad$ $\sqrt[n]{x^n} = |x|$ when n is even

For this course, it is possible to leave final answers with radical denominators (unless you are instructed to rationalize the denominator).

$\frac{1}{\sqrt{2}}$ ✅ $\qquad\qquad$ $\sqrt{\frac{1}{3}}$ ❌ but $\sqrt{\frac{1}{3}} = \frac{\sqrt{1}}{\sqrt{3}} = \frac{1}{\sqrt{3}}$ ✅

Reviewing Radical Operations

Ex. Simplify. Rationalize the denominator.

a. $\frac{3\sqrt{6}}{4\sqrt{3}}$

$= \frac{3\sqrt{6}}{4\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$

$= \frac{3\sqrt{18}}{4\times 3}$

$= \frac{3\sqrt{2}}{4}$

b. $\frac{1}{\sqrt{2}+3}$

$= \frac{1}{\sqrt{2}+3} \times \frac{\sqrt{2}-3}{\sqrt{2}-3}$

$= \frac{\sqrt{2}-3}{2-9}$

$= -\frac{\sqrt{2}-3}{7}$ or $\frac{3-\sqrt{2}}{7}$

c. $\sqrt{\frac{2}{3}}$

$= \frac{\sqrt{2}}{\sqrt{3}}$

$= \frac{\sqrt{6}}{3}$

d. $\frac{1}{\sqrt[3]{x}}$

$= \frac{1}{\sqrt[3]{x}} \cdot \frac{\sqrt[3]{x^2}}{\sqrt[3]{x^2}}$

$= \frac{\sqrt[3]{x^2}}{x}$

Challenge:

e. $\frac{1}{\sqrt[3]{x}-2}$

f. $\frac{1}{\sqrt[3]{x^2}-\sqrt[3]{x}+1}$

Hint: $a^3 - b^3 = (a-b)(a^2+ab+b^2)$

$a^3 + b^3 = (a+b)(a^2-ab+b^2)$

$= \frac{1}{\sqrt[3]{x}-2} \cdot \frac{\left(\sqrt[3]{x}\right)^2+2\sqrt[3]{x}+4}{\left(\sqrt[3]{x}\right)^2+2\sqrt[3]{x}+4}$

$= \frac{\sqrt[3]{x^2}+2\sqrt[3]{x}+4}{x-8}$

$= \frac{1}{\sqrt[3]{x^2}-\sqrt[3]{x}+1} \cdot \frac{\sqrt[3]{x}+1}{\sqrt[3]{x}+1}$

$= \frac{\sqrt[3]{x}+1}{x+1}$

Review Solving Equations Using Radicals

Square Root Principle

The square root principle states that if a squared variable equals a number ($x^2 = n$), then the variable equals both the positive and negative square roots of that number ($x = \pm\sqrt{n}$)

Quadratic Formula

For some $ax^2 + bx + c = 0$

$$\boxed{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

Ex. Solve for x.

a. $x^2 = 40$

$x = \pm\sqrt{40}$

$x = \pm 2\sqrt{10}$

b. $x^3 = 64$

$x = \sqrt[3]{64}$

$x = 4$

c. $(2x - 1)^2 - 15 = 0$

$(2x - 1)^2 = 15$

$2x - 1 = \pm\sqrt{15}$

$2x = 1 \pm \sqrt{15}$

$x = \frac{1 \pm \sqrt{15}}{2}$

d. $3x^2 - 6x - 2 = 0$

$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(3)(-2)}}{2(3)}$

$x = \frac{6 \pm \sqrt{60}}{6}$

$x = \frac{6 \pm 2\sqrt{15}}{6}$

$x = \frac{3 \pm \sqrt{15}}{3}$

Radical Graphs, $y = a\sqrt{b(x - c)} + d$

In this chapter, the main focus will be the square root function and the possible transformations that can be applied to it.

As with all graphs we have seen thus far, radical graphs are subject to the same transformation principles covered in previous sections.

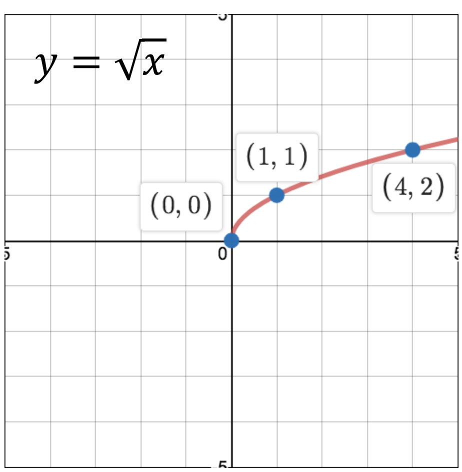
$a -$ v. exp / comp and v. reflection (over x-axis)

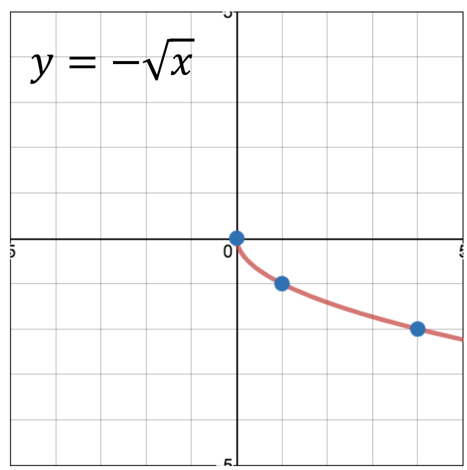
$b -$ h. exp / comp and h. reflection (over y-axis)

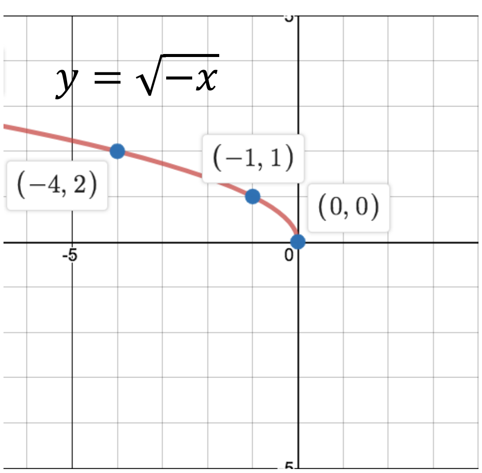
$c -$ h. translation left or right

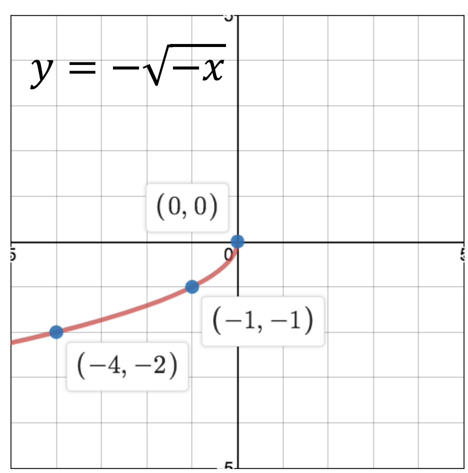
$d -$ v. translation up or down

Graph of $y = \sqrt{x}$ and Reflections









Radical Functions and Transformations

Horizontal Translations, c

$$\boxed{y = \sqrt{x - c}}$$

Ex. For the following, state the transformation of the $y = \sqrt{x}$.

a. $y = \sqrt{x - 4}$

h. translation 4 units to the right

b. $y = \sqrt{x + 3}$

h. translation 3 units to the left

Vertical Translations, d

$$\boxed{y = \sqrt{x} + d}$$

Ex. For the following, state the transformation of the $y = \sqrt{x}$.

a. $y = \sqrt{x} + 5$

v. translation 5 units up

b. $y = \sqrt{x} - 2$

v. translation 2 units down

Vertical Expansion/Compression and Reflection, a

$$\boxed{y = a\sqrt{x}}$$

Ex. For the following, state the transformation of the $y = \sqrt{x}$.

a. $y = 2\sqrt{x}$

v. exp by a factor of 2

b. $y = -\frac{1}{2}\sqrt{x}$

v. comp by a factor of $\frac{1}{2}$
reflection over x-axis

Horizontal Expansion/Compression and Reflection, b

$$\boxed{y = \sqrt{bx}}$$

Ex. For the following, state the transformation of the $y = \sqrt{x}$.

a. $y = \sqrt{-3x}$

h. comp by a factor of $\frac{1}{3}$
reflection over y-axis

b. $y = \sqrt{\frac{2}{3}x}$

v. exp by a factor of $\frac{3}{2}$

Graphing $y = a\sqrt{b(x-c)} + d$ Using Mapping Notation

$y = a\sqrt{b(x-c)} + d$	$(x, y) \to \left(\frac{1}{b}x + c, ay + d\right)$

Ex. Graph $f(x) = 2\sqrt{-(x+1)} + 3$

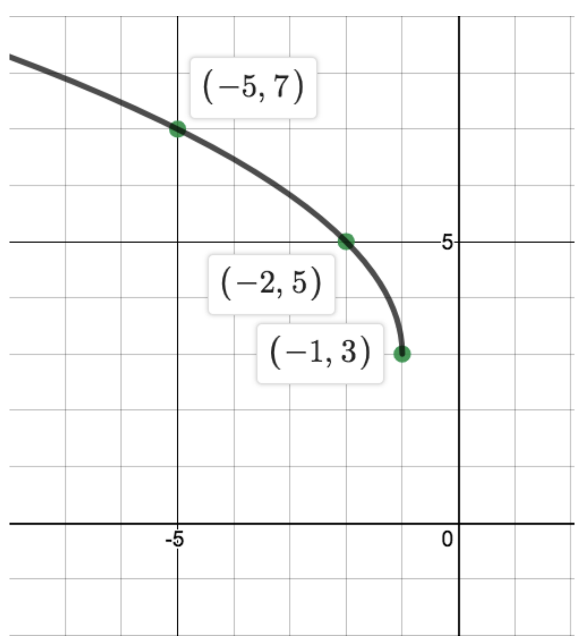
Use mapping notation to graph $f(x)$
$(x, y) \to (-x - 1, 2y + 3)$

Find coordinates for the base function $y = \sqrt{x}$
$(0,0)$, $(1,1)$, $(4,2)$, $(9,3)$

Mapping:
$(0,0) \to (-0 - 1, 2(0) + 3)$
$= (-1, 3)$

$(1,1) \to (-1 - 1, 2(1) + 3)$
$= (-2, 5)$

$(4,2) \to (-4 - 1, 2(2) + 3)$
$= (-5, 7)$



Graphing Radical Functions with Even and Odd Index Numbers

Odd Index Number $\boldsymbol{n}$ for $y = \sqrt[n]{x}$

The domain and range is all real numbers

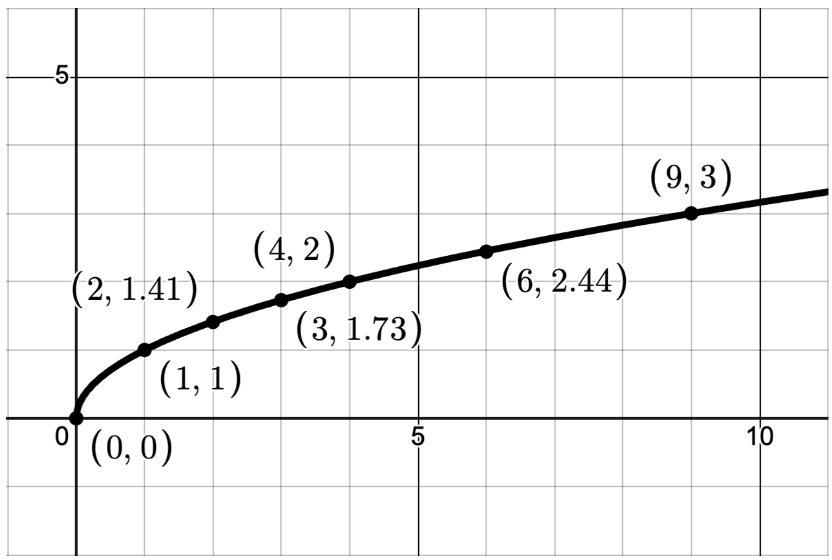
Even Index Number $\boldsymbol{n}$ for $y = \sqrt[n]{x}$

The domain is restricted to $x \geq 0$ and the range is $y \geq 0$. These could change depending on transformations.

Ex. Complete the table of values to graph the following.

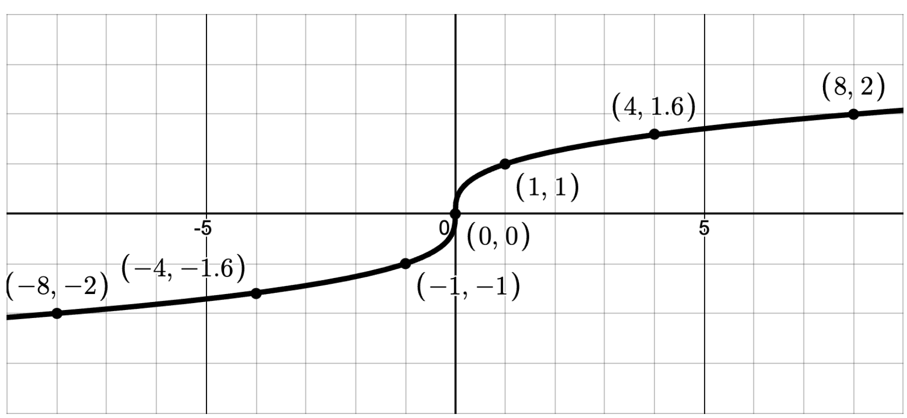
a. $y = \sqrt{x}$

x	y
0	0
1	1
2	1.41
3	1.73
4	2
6	2.44
9	3



b. $y = \sqrt[3]{x}$

x	y
-8	-2
-4	-1.6
-1	-1
0	0
1	1
4	1.6
8	2

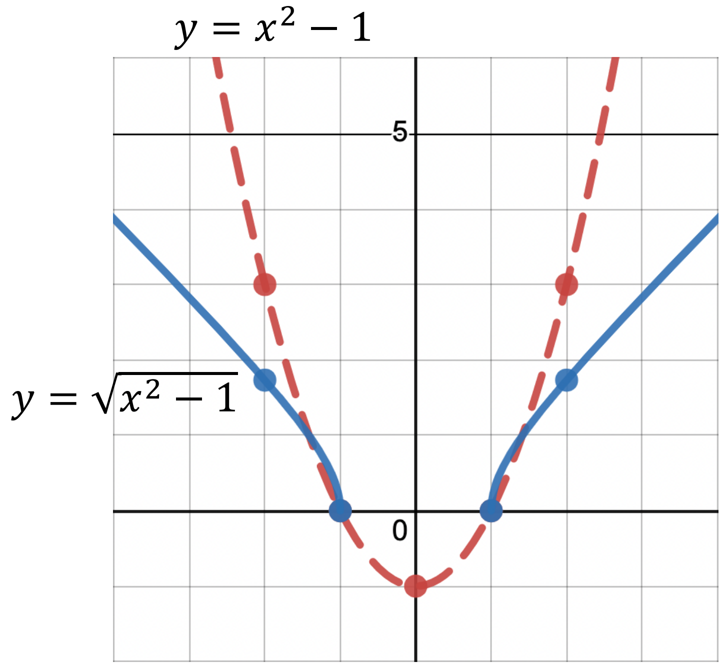


Applying Radicals as a Transformation to Another Function $y = \sqrt{f(x)}$

First, complete a table of values for $y = f(x)$. Then, apply a square to all the y values. Then graph $y = \sqrt{f(x)}$.

Ex. Graph $y = \sqrt{x^2 - 1}$

x	$x^2 - 1$	$\sqrt{x^2 - 1}$
-3	8	2.8
-2	3	1.7
-1	0	0
0	-1	not real
1	0	0
2	3	1.7
3	8	2.8



4.1 Homework

#2 bcf…, 4 bcf…, 5 abdfgijmn, 6 abcd

4.2 Graphing and Solving Radicals

Solve by Graphing

Ex. Solve for the x-int for $f(x) = \sqrt{5-x} - 2$

To find x-int, we graph the function, and determine where it meets the x-axis.

$$y = \sqrt{-(x-5)} - 2$$

Transformations: h. reflection
h. translation 5 to the right
v. translation 2 down

Mapping:

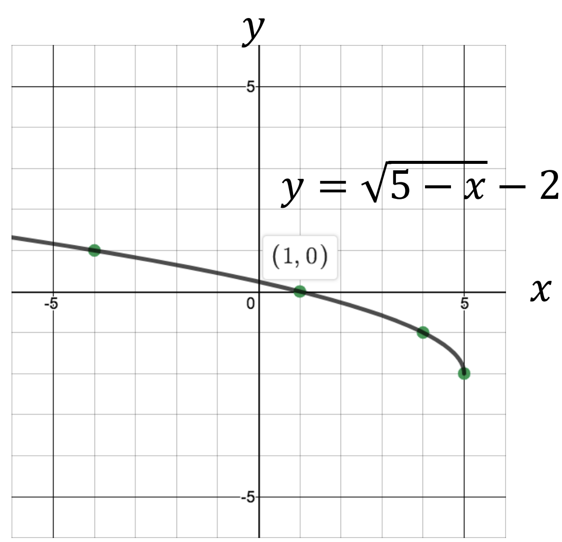
$(x, y) \rightarrow (-x+5, y-2)$

$(0, 0) \rightarrow (5, -2)$

$(1, 1) \rightarrow (4, -1)$

$(4, 2) \rightarrow (1,0)$

$(9, 3) \rightarrow (-4,1)$



$\therefore x - \text{int} = 1$ or $(1,0)$

Now, solve algebraically to confirm solution.

$\sqrt{5-x} - 2 = 0$

$\sqrt{5-x} = 2$

* square both sides

$5 - x = 4$

$-x = 4 - 5$

$-x = -1$

$x = 1$ * check your answer

Ex. Solve $\sqrt{x+6} - x = 0$

Solving graphically:

Separate the radical portion, from the rest:

$$\sqrt{x+6} = x$$

Instead of finding the x-int of the original function, determine the points of intersection between the two functions: $y = \sqrt{x+6}$ and $y = x$

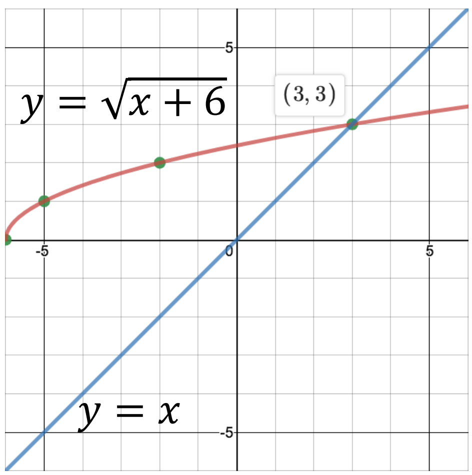
$y = \sqrt{x+6},$

h. translation 6 to the left

Mapping: $(x, y) \rightarrow (x-6, y)$

$(0,0) \rightarrow (-6,0)$ $\quad$ $(1,1) \rightarrow (-5,1)$

$(4,2) \rightarrow (-2,2)$ $\quad$ $(9,3) \rightarrow (3,3)$



The point of intersection is $(3, 3)$

$\therefore x = 3$

Ex. Solve $\sqrt{x+6} - x = 0$ (algebraically)

Just like the graphing step, separate the radical from the rest

$$\sqrt{x+6} = x$$

* square both sides

$$x + 6 = x^2$$

* make one side of the equation equal to 0

$$0 = x^2 - x - 6$$

$$0 = (x-3)(x+2)$$

$$x = -2, 3$$

* check your answer

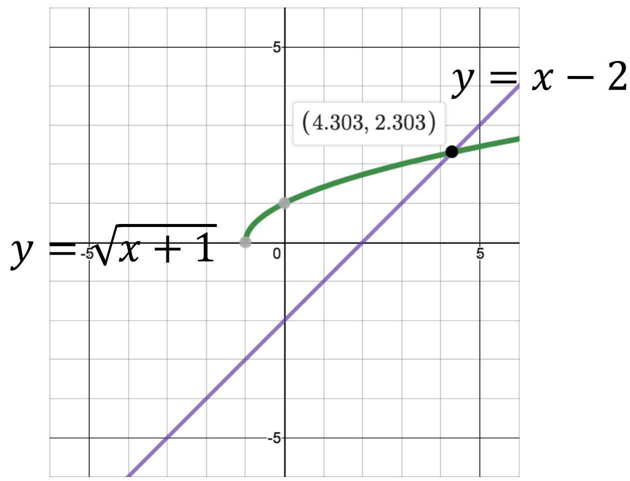
Reject -2, it is extraneous

$$x = 3$$

Ex. Solve $\sqrt{x+1} - x + 2 = 0$

$$\sqrt{x+1} = x - 2$$

Graphically



$\therefore x \approx 4.303$

Algebraically

$\sqrt{x+1} = x - 2$

$x + 1 = (x-2)^2$

$x + 1 = x^2 - 4x + 4$

$0 = x^2 - 5x + 3$

$x = \frac{5 \pm \sqrt{25-12}}{2}$

$x = \frac{5+\sqrt{13}}{2}, \frac{5-\sqrt{13}}{2}$

Reject $x = \frac{5-\sqrt{13}}{2}$,

it is extraneous

$x = \frac{5+\sqrt{13}}{2}$

Ex. Solve $2\sqrt{x+3} - \sqrt{2x-1} = 0$

$2\sqrt{x+3} = \sqrt{2x-1}$

$\left(2\sqrt{x+3}\right)^2 = \left(\sqrt{2x-1}\right)^2$

$4(x+3) = 2x - 1$

$4x + 12 = 2x - 1$

$2x = -13$

$x = -\frac{13}{2}$

Reject $x = -\frac{13}{2}$

$\therefore$ No real solutions

Radical Equations with 2 Radicals and a Constant

Ex. Solve $3\sqrt{x+2} + \sqrt{6x+1} - 2 = 0$

*separate the two radicals on either side; the left over 2 can go either side of the equation, I just choose to put it on the right side

$$3\sqrt{x+2} = 2 - \sqrt{6x+1}$$
$$\left(3\sqrt{x+2}\right)^2 = \left(2 - \sqrt{6x+1}\right)^2$$
$$9(x+2) = \left(2 - \sqrt{6x+1}\right)\left(2 - \sqrt{6x+1}\right)$$

*use distributive property

$$9x + 18 = 4 - 2\sqrt{6x+1} - 2\sqrt{6x+1} + 6x + 1$$
$$9x + 18 = 5 - 4\sqrt{6x+1} + 6x$$

* separate radical from the rest

$$3x + 13 = -4\sqrt{6x+1}$$
$$(3x+13)^2 = \left(-4\sqrt{6x+1}\right)^2$$

$$9x^2 + 39x + 39x + 169 = 16(6x+1)$$
$$9x^2 + 78x + 169 = 96x + 16$$
$$9x^2 - 18x + 153 = 0$$
$$x^2 - 2x + 17 = 0$$
$$x = \frac{2 \pm \sqrt{4-68}}{2}$$
$$= \frac{2 \pm \sqrt{-64}}{2}$$

Optional: $= \frac{2 \pm 8i}{2}$
$= 1 + 4i, 1 - 4i$

$\therefore$ no real solutions

4.2 Homework:

2ab, 3-4 bcf…, 5 adf, 6 ace

4.3 Rational Functions

Rational Functions

$f(x)$ is a **Rational Function** if:

$$\boxed{f(x) = \frac{g(x)}{h(x)}}$$

where $g(x)$ and $h(x)$ are polynomial functions and $h(x) \neq 0$

In general, rational functions look like $f(x) = \frac{ax^m + \cdots}{bx^n + \cdots}$

Examples of Rational Functions

$f(x) = \frac{1}{x}$ $\qquad$ $f(x) = \frac{3x+1}{2x-4}$

$f(x) = \frac{3x^2+5}{4}$ $\qquad$ $f(x) = \frac{x^2}{x^3-1}$

Examples of Non-Rational Functions

$f(x) = \frac{1}{\sqrt{x}}$ $\qquad$ $f(x) = \frac{x^2-1}{2^x}$

Non-Permissible Values (NPV)

For $f(x) = \frac{g(x)}{h(x)}$, NPVs at the roots of $h(x)$.

Ex. Determine the non-permissible values for the following function

$f(x) = \frac{5x+2}{6x^2+7x+2}$

$$6x^2 + 7x + 2 = 0$$
$$(3x + 2)(2x + 1) = 0$$
$$x = -\frac{2}{3}, -\frac{1}{2}$$

$\therefore$ there are NPVs at $x = -\frac{2}{3}$ and $x = -\frac{1}{2}$

Asymptotes and Points of Discontinuity (Hole)

Ex. For $f(x) = \frac{x^2+3x+2}{x^2-4}$, simplify and determine all asymptotes and points of discontinuity (holes).

First, factor $g(x)$ and $h(x)$, state NPVs and simplify

$$f(x) = \frac{(x+1)(x+2)}{(x+2)(x-2)}$$

$$f(x) = \frac{x+1}{x-2} \qquad \text{, NPVs:} \qquad x \neq -2, 2$$

Vertical Asymptote: * NPV that is still "visible" in simplified $f(x)$

$$x = 2$$

Hole: the point where the graph has a gap

$$x = -2$$

To get the y coordinate, plug in $x = -2$ into the simplified $f(x)$

$$f(-2) = \frac{-2+1}{-2-2}$$

$$= \frac{-1}{-4} = \frac{1}{4}$$

Hole: $\left(-2, \frac{1}{4}\right)$

The point of discontinuity and vertical asymptote have been identified, but there is still one more piece of information missing: horizontal asymptote.

H. Asymptote

These asymptotes are similar to vertical asymptotes, but they are considered pseudo-asymptotes because the curve can cross horizontal asymptotes.

To find horizonal asymptotes, determine which of the three scenarios does it belong to.

For $f(x) = \frac{ax^m+\cdots}{bx^n+\cdots}$

1) $m = n$

   HA: $y = \frac{a}{b}$

2) $m > n$

   HA: none

3) $m < n$

   HA: $y = 0$

Find horizontal asymptote for $f(x) = \frac{x+1}{x-2}$

$f(x) = \frac{x+1}{x-2}$, this is scenario (1)

HA: $y = \frac{1}{1} = 1$

$y = 1$

Alternatively, it is possible to find HA algebraically:

$$f(x) = \frac{x+1}{x-2}$$

$$= \frac{x-2+1+2}{x-2}$$

$$= \frac{x-2+3}{x-2}$$

$$= \frac{x-2}{x-2} + \frac{3}{x-2}$$

$$= \frac{3}{x-2} + 1$$

∴ the complete solution for the previous question:

VA: $x = 2$ HA: $y = 1$ Hole: $\left(-2, \frac{1}{4}\right)$

Ex. Simplify and determine all asymptote(s) and hole(s) for

$$f(x) = \frac{x^2-2x-24}{3x^2-48}$$

$$f(x) = \frac{(x+4)(x-6)}{3(x+4)(x-4)}$$

$$f(x) = \frac{x-6}{3(x-4)}$$

VA: $x = 4$ HA: $y = \frac{1}{3}$

Holes: $x = -4$

$$f(-4) = \frac{-4-6}{3(-4-4)} = \frac{-10}{-24} = \frac{5}{12}$$

$$\left(-4, \frac{5}{12}\right)$$

Ex. Determine all asymptote(s) and hole(s) $f(x) = \frac{x+1}{x^2-5x-6}$

$$f(x) = \frac{x+1}{(x+1)(x-6)}$$

$$f(x) = \frac{1}{x-6}$$

VA: $x = 6$ HA: $y = 0$

Holes: $\left(-1, -\frac{1}{7}\right)$ $f(-1) = \frac{1}{-1-6} = -\frac{1}{7}$

Ex. Determine all asymptotes and holes $f(x) = \frac{x^2+2x}{x-3}$

$$f(x) = \frac{x(x+2)}{x-3}$$

VA: $x = 3$ HA: none*

Holes: none

* there are no HA because $m > n$,
but if $m = n + 1$, there is an oblique asymptote (slant asymptote)

To find SA (slant asymptote):

SA = $g(x) \div h(x)$

→ SA is the quotient of $g(x) \div h(x)$

```
3   |   1   2   0
    |       3   15
-------------------------
        1   5 | 15
```

∴ SA: $y = x + 5$

Ex. Determine the x-int and y-int.

a. $f(x) = \frac{x^2-9}{x^2-x-2}$

$= \frac{(x+3)(x-3)}{(x+1)(x-2)}$

x-int:

$(x+3)(x-3) = 0$

$x = -3, 3 \quad \rightarrow (-3,0), (3,0)$

y-int: $y = \frac{-9}{-2} = \frac{9}{2} \quad \rightarrow \left(0, \frac{9}{2}\right)$

b. $f(x) = \frac{1}{x+1} - \frac{1}{x-1} + 2$

x-int:

$0 = \frac{1}{x+1} - \frac{1}{x-1} + 2$

*multiply by LCD = $(x+1)(x-1)$

$0 = 1(x-1) - 1(x+1) + 2(x+1)(x-1)$

$0 = x - 1 - x - 1 + 2x^2 - 2$

$0 = 2x^2 - 4$

$2x^2 = 4$

$x^2 = 2$

$x = -\sqrt{2}, \sqrt{2} \quad \rightarrow \left(-\sqrt{2}, 0\right), \left(\sqrt{2}, 0\right)$

y-int:

$y = \frac{1}{1} - \frac{1}{-1} + 2 = 4 \quad \rightarrow (0, 4)$

4.3 Homework

2-4 bcf…

4.4 Graphing Rational Functions

To graph $f(x) = \frac{g(x)}{h(x)}$

1. factor $g(x)$ and $h(x)$, then simplify
2. determine the VA, HA, Holes, or SA
3. determine x and y intercepts (if possible)
4. create a table of values (one for each "section")
5. plot the coordinates and connect the points with a smooth curve
6. label all asymptotes, axes, scales

Ex. Graph $y = \frac{2x^2}{x^2+1}$

Identify all the asymptote and point(s) of discontinuity

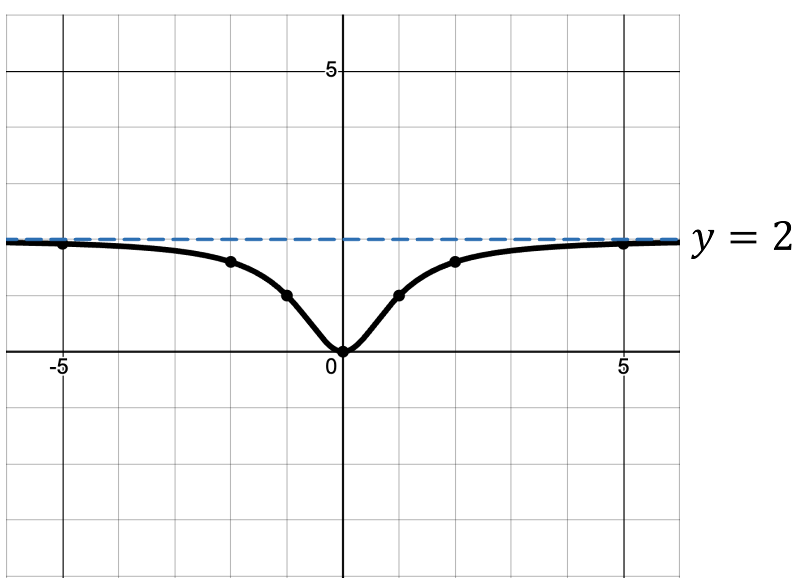
HA: $y = 2$

State all x and y intercepts

x-int and y-int: $(0,0)$

Create a table values, make sure to include enough coordinates to

x	-5	-2	-1	0	1	2	5
y	1.9	1.6	1	0	1	1.6	1.9



Ex. Graph $y = \frac{2}{x+1}$

First, state all the asymptote and point(s) of discontinuity, as well x and y intercepts

VA: $x = -1$ HA: $y = 0$

x-int: none y-int: $(0, 2)$

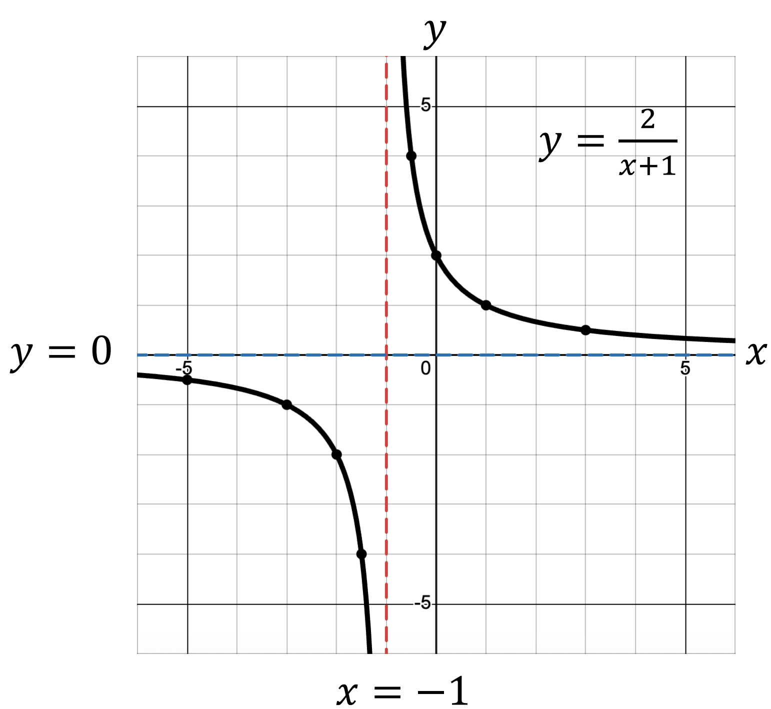
Create a table for each section separated by vertical asymptote(s)

Left section:
(Left of $x = -1$)

x	-5	-3	-2	-1.5	-1.25
y	-0.5	-1	-2	-4	-8

Right section:
(Right of $x = -1$)

x	-0.75	-0.5	0	1	3
y	8	4	2	1	0.5



Ex. Graph $y = \frac{x^2+x-2}{2+x-x^2}$

First factor the function and simplify if possible

$$y = \frac{x^2+x-2}{-x^2+x+2} = -\frac{(x+2)(x-1)}{(x-2)(x+1)}$$

State all the asymptote and point(s) of discontinuity

VA: $x = 2, -1$ HA: $y = \frac{1}{-1} = -1$

x-int: $(-2, 0), (1,0)$ y-int: $(0, -1)$

Left section (left of $x = -1$)

x	-6	-3	-2	-1.5	-1.25
y	-0.7	-0.4	0	0.7	2.1

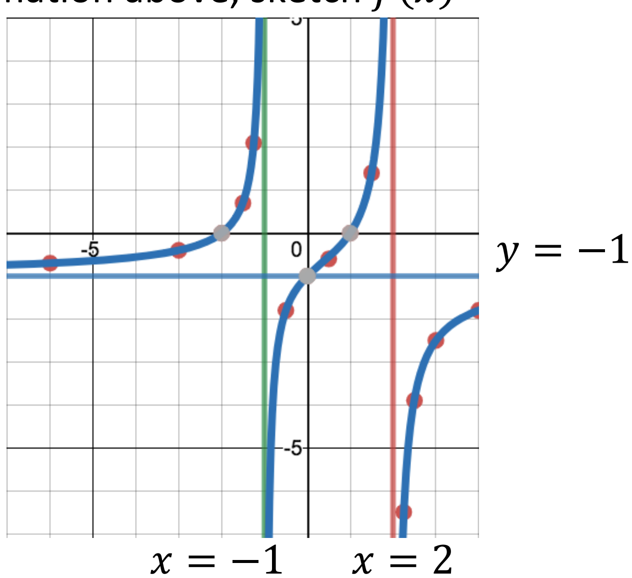
Middle section (in between -1 and 2)

x	-0.75	-0.5	0.5	1.5	1.75
y	-3.2	-1.8	-0.6	1.4	4.1

Right section (right of $x = 2$)

x	2.25	2.5	3	4	5
y	-6.5	-3.9	-2.5	-1.8	-1.6

With the information above, sketch $f(x)$



Ex. Graph $y = \frac{x^2 - x - 2}{x^2 - 4}$

$$y = \frac{(x-2)(x+1)}{(x+2)(x-2)}$$

$y = \frac{x+1}{x+2}$ $x - 2$ is cancelled, there will be a hole at $x = 2$

VA: $x = -2$ Holes: $\left(2, \frac{3}{4}\right)$

HA: $y = 1$

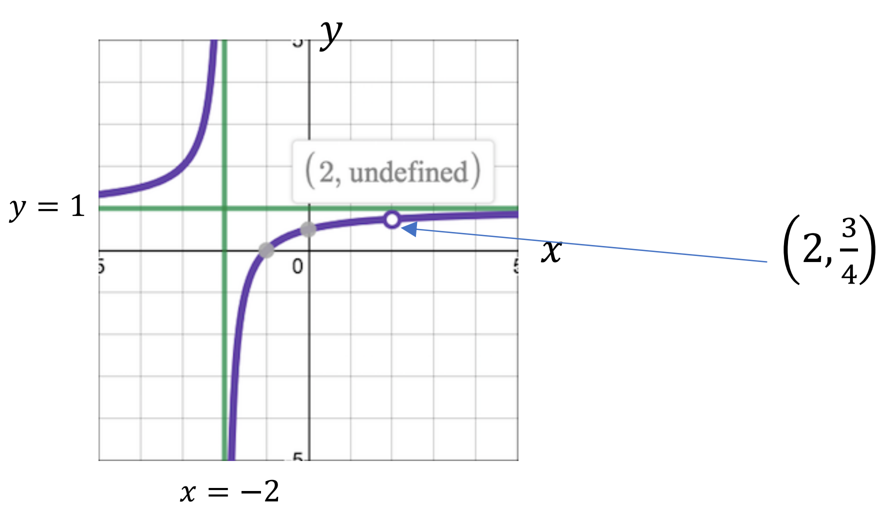
x-int: $(-1, 0)$ y-int: $\left(0, \frac{1}{2}\right)$

Left side (Left of $x = -2$)

x	-6	-4	-3	-2.5	-2.25
y	1.25	1.5	2	3	5

Right side (Right of $x = -2$)

x	-1.75	-1.5	-1	0	2
y	-3	-1	0	0.5	0.75



4.4 Homework

1-6 bcf…