Chapter 2 – Limits and Continuity

2.1 Limits

Limits are used to describe the behaviour of a function as it approaches a specific value; how it behaves near at a point, instead of at the point.

Limit Notation:	
Left-side limit:	Right-side limit:
Limit of $f(x)$ as x approaches a:	

Suppose that $f(x)$ approaches a single number L, as x approaches a from both sides.

$$\lim_{x \to a} f(x) = L$$

The limit of $f(x)$, as x approaches a, equals L.

The limit of a function $f(x)$ exists at $x = a$ if and only if both one-sided limits exist and have the same value of L.

So,

or else

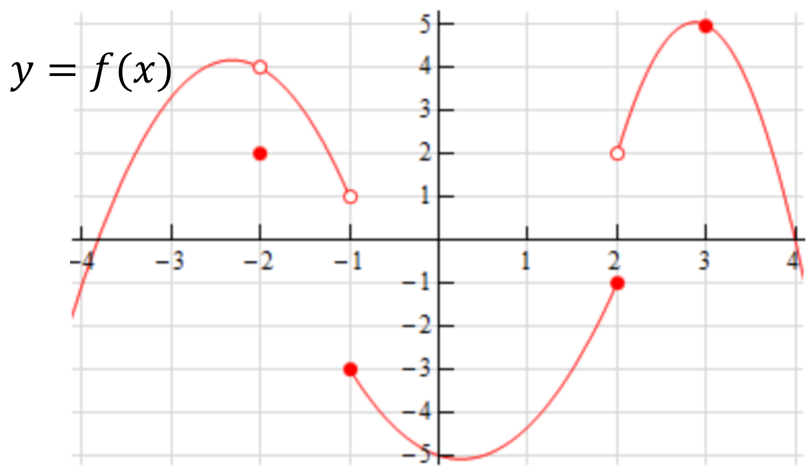
Ex. Given $\lim_{x \to 2^-} f(x) = 3$ and $\lim_{x \to 2^+} f(x) = 3$, determine $\lim_{x \to 2} f(x)$.

The left-side limit and right-side limit are the same, so the limit must evaluate to be the same value.

Ex. Given $\lim_{x \to a^-} f(x) = 5$ and $\lim_{x \to a^-} f(x) = -3$, determine $\lim_{x \to a} f(x)$.

The limit does not exist when the left-side and right-side are not equal.

Ex. Use the graph to evaluate the following limits.



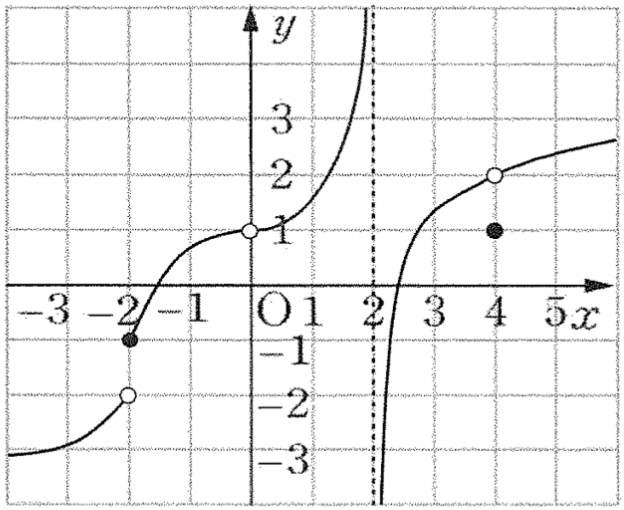
a. $\lim_{x \to -2^-} f(x)$ b. $\lim_{x \to -2^+} f(x)$ c. $\lim_{x \to -2} f(x)$ d. $f(-2)$

e. $\lim_{x \to -1^-} f(x)$ f. $\lim_{x \to -1^+} f(x)$ g. $\lim_{x \to -1} f(x)$ h. $f(-1)$

i. $\lim_{x \to 2^-} f(x)$ j. $\lim_{x \to 2^+} f(x)$ k. $\lim_{x \to 2} f(x)$ l. $f(2)$

m. $\lim_{x \to 3^-} f(x)$ n. $\lim_{x \to 3^+} f(x)$ o. $\lim_{x \to 3} f(x)$ p. $f(3)$

Ex. Use the graph of f to evaluate the following.



a. $\lim_{x \to -2^-} f(x) =$

b. $\lim_{x \to -2^+} f(x) =$

c. $\lim_{x \to -2} f(x) =$

d. $f(-2) =$

e. $\lim_{x \to 0^-} f(x) =$

f. $\lim_{x \to 0^+} f(x) =$

g. $\lim_{x \to 0} f(x) =$

h. $f(0) =$

i. $\lim_{x \to 2^-} f(x) =$

j. $\lim_{x \to 2^+} f(x) =$

k. $\lim_{x \to 2} f(x) =$

l. $f(2) =$

m. $\lim_{x \to 4^-} f(x) =$

n. $\lim_{x \to 4^+} f(x) =$

o. $\lim_{x \to 4} f(x) =$

p. $f(4) =$

Properties of Limits

Assume $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ both exist and c is a constant.

1. $\left[\lim_{x \to a} cf(x)\right] =$

2. $\lim_{x \to a}[f(x) \pm g(x)] =$

3. $\lim_{x \to a}[f(x)g(x)] =$

4. $\lim_{x \to a}\left[\frac{f(x)}{g(x)}\right] =$

5. $\lim_{x \to a}[f(x)]^n =$

6. $\lim_{x \to a}\sqrt[n]{f(x)} =$

Ex. Use the properties of limits to evaluate the following limit.

$$\lim_{x \to 2}\frac{2x^3 - x^2 + 4}{\sqrt{x+2}}$$

Ex. Given $\lim_{x \to 2} f(x) = 3$, $\lim_{x \to 2} g(x) = 10$, and $\lim_{x \to 2} h(x) = -5$ use the limit properties to compute each of the following limits. If not possible, explain.

a. $\lim_{x \to 2} \left[\frac{f(x)}{g(x)} - \frac{h(x)}{f(x)} \right]$

b. $\lim_{x \to 2} [f(x)g(x)h(x)]$

c. $\lim_{x \to 2} \left[\frac{1}{h(x)} + \frac{2 - f(x)}{g(x) + h(x)} \right]$

d. $\lim_{x \to 2} \left[2f(x) - \frac{1}{g(x)+2h(x)} \right]$

Evaluating Limits Using Table of Values and Graphs

Ex. Given $f(x) = \begin{cases} \dfrac{2x-2}{x^2-1} & , x \neq 1 \\ 3 & , x = 1 \end{cases}$, find the value of $\lim_{x \to 1} f(x)$.

Table of Values:

x	0.9	0.99	0.999	0.9999	1	1.0001	1.001	1.01	1.1
y	1.0526	1.00503	1.00050	1.00005	**3**	0.99995	0.99950	0.99502	0.95238

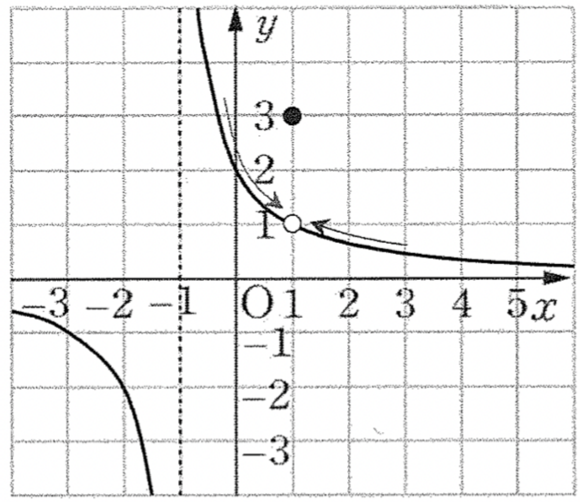
$\lim_{x \to 1^-} f(x) = 1$

(Left-hand limit)

$\lim_{x \to 1^+} f(x) = 1$

(Right-hand limit)

Graphically:



Ex. Given $f(x) = \begin{cases} 2-x & , \ x \le 2 \\ x & , \ x > 2 \end{cases}$, find the value of $\lim_{x \to 2} f(x)$.

Table of Values:

x	1.9	1.99	1.999	1.9999	2	2.0001	2.001	2.01	2.1
y	0.1	0.01	0.001	0.0001	**0**	2.0001	2.001	2.01	2.1

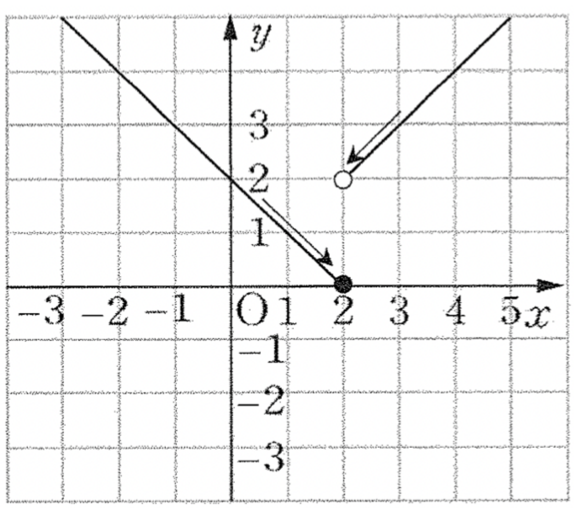
$\lim_{x \to 2^-} f(x) = 0$

(Left-hand limit)

$\lim_{x \to 2^+} f(x) = 2$

(Right-hand limit)

Graphically:



Evaluating Limits Algebraically

There are multiple techniques to try and use when evaluating limits; the most straight-forward method is direct substitution.

Evaluating limits using Direct Substitution Property

If $f(x)$ is a polynomial or rational function and a is in the domain of $f(x)$:

$\lim_{x \to a} f(x) =$ substitute in a into $f(x)$ and evaluate

Ex. Find the following limits.

a. $\lim_{x \to 2}(3x^2 + x - 2)$

b. $\lim_{x \to 0} \frac{6x^2+2x-7}{x+3}$

If using Direct Substitution Property results in an indeterminate solution $\frac{0}{0}$, try using other methods.

Limits With Rational Expressions using Factoring

1. Factor the numerator and denominator
2. Cancel the common factor

Ex. Given $f(x) = \frac{x^2-4x+3}{x-3}$, find the value of $\lim_{x \to 3} f(x)$.

Ex. Evaluate the following limits.

a. $\lim_{x \to 1} \frac{x^2 - x}{x - 1}$

b. $\lim_{x \to 3} \frac{x^2 - 9}{x^3 - 27}$

c. $\lim_{x \to -5} \frac{x^3 + 125}{x + 5}$

Ex. Given $f(x)=\begin{cases}\dfrac{2x-2}{x^2-1}, & x\neq 1\\ 3, & x=1\end{cases}$, find the value of $\lim_{x\to 1} f(x)$.

For this piece-wise function, the function is the same before and after $x=1$, so there is no need to evaluate both one-sided limits.

Limits With Radical Expressions using Conjugate Pairs

1. Multiply the numerator and denominator by the conjugate
2. Cancel the common factor

Ex. Evaluate the following limits.

a. $\lim_{x\to 0}\frac{\sqrt{4+x}-2}{x}$

b. $\lim_{x \to 1} \frac{\sqrt{x}-1}{x-1}$

Alternatively, we could factor…but how?

Limits With Complex Fractions using LCD

1. Use lowest common denominator to combine fractions
2. Cancel the common factor

Ex. Evaluate the following limits.

a. $\lim_{x \to 3} \frac{\frac{1}{3}-\frac{1}{x}}{x-3}$

b. $\lim_{x \to 0} \frac{\frac{1}{x+2} - \frac{1}{2}}{x}$

Limits with Piece-wise Functions

Since the function can be different on either side of a value a, need to value both one-sided limits separately.

$\lim_{x \to a} f(x)$ where $f(x) = \begin{cases} g(x), x < a \\ h(x), x \geq a \end{cases}$

Need to evaluate both $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$.

Ex. Given $f(x) = \begin{cases} 6x, & x \leq -4 \\ 1 - 9x, & x > -4 \end{cases}$. Evaluate the following.

a. $\lim_{x \to 7} f(x)$

b. $\lim_{x \to -4} f(x)$

Ex. Given $f(x)=\begin{cases} 2-x, & x\le 2 \\ x, & x>2 \end{cases}$, find the value of $\lim_{x\to 2} f(x)$.

Ex. Let $f(x)=\begin{cases} 3x^2-2x+1, & x<1 \\ x^3+1, & x\ge 1 \end{cases}$. Find $\lim_{x\to 1} f(x)$.

Limits with Absolute Value

Definition of Absolute Value (as a Piece-wise Function)

$$|x|=\begin{cases} -x & \text{if } x<0 \\ x & \text{if } x\ge 0 \end{cases}$$

Evaluating limits with absolute value is very similar to evaluating limits with piece-wise function.

Ex. Evaluate the following limits.

a. $\lim_{t \to 2} \frac{2-t}{|t-2|}$

b. $\lim_{x \to 4} \frac{|x-4|}{x^2-16}$

Ex. Show that $\lim_{x \to 0} \frac{|x|}{x}$ does not exist.

Limits of Reciprocal Functions at 0

$\lim_{x \to 0^-} \frac{1}{x} =$ $\lim_{x \to 0^+} \frac{1}{x} =$ $\lim_{x \to 0} \frac{1}{x} =$

$\lim_{x \to 0^-} \frac{1}{x^2} =$ $\lim_{x \to 0^+} \frac{1}{x^2} =$ $\lim_{x \to 0} \frac{1}{x^2} =$

Ex. Evaluate the following limits.

a. $\lim_{x \to 1^-} \frac{4}{x-1}$

b. $\lim_{x \to 1^+} \frac{4}{x-1}$

c. $\lim_{x \to 1} \frac{4}{x-1}$

d. $\lim_{x \to -1^+} \frac{x}{x+1}$

e. $\lim_{x \to 1^-} \frac{x-1}{|x-1|}$

Greatest Integer Function (Floor Function) $\lfloor x \rfloor$ or $[x]$

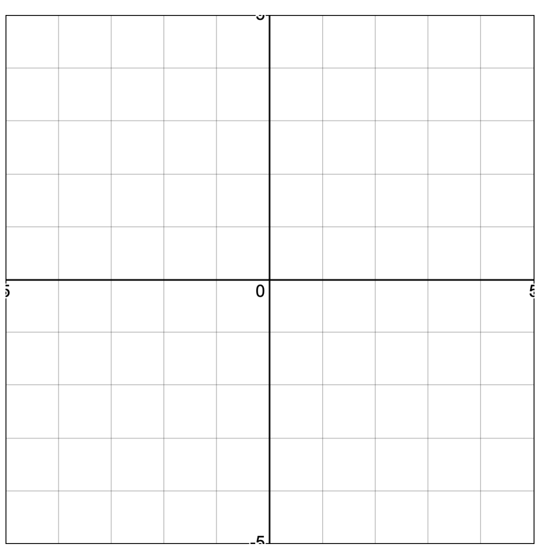
$\lfloor x \rfloor$ or $\operatorname{int}(x)$ represents the greatest integer less than or equal to x
$\lfloor x \rfloor = n$ if and only if $n \leq x < n + 1$ where $n \in \mathbb{Z}$

$\lfloor 3.9 \rfloor =$ $\quad$ $\lfloor 2.1 \rfloor =$ $\quad$ $\lfloor -1.8 \rfloor =$ $\quad$ $\lfloor -5.1 \rfloor =$

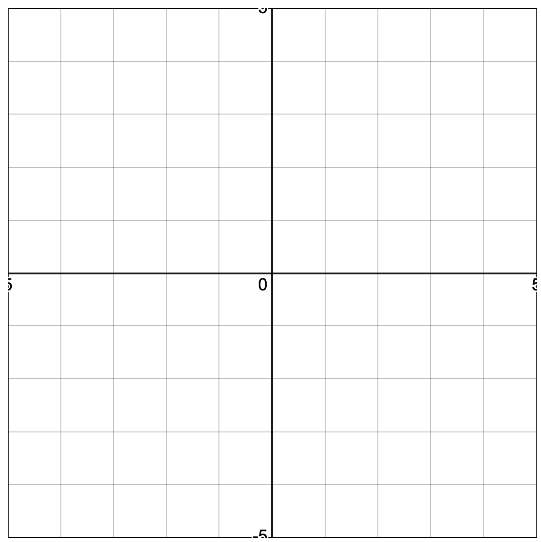
Ex. Show that $\lim_{x \to 0} \lfloor x \rfloor$ does not exist

x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
y							

Ex. Sketch the graph of $y = \lfloor x \rfloor$ on the interval $-3 \leq x < 3$ and show that $\lim_{x \to 0} \lfloor x \rfloor$ does not exist.



Ex. Sketch the graph of $y = \frac{1}{x}$ and show that $\lim_{x \to 0} \frac{1}{x}$ does not exist.



2.1 Homework:

Limits Worksheet

Stewart:

pg 96 # 2, 3, 5, 7, 9, 11, 23, 26, 29, 32, 33, 37, 38a

pg 106 # 1, 3, 11-32, 41-46,48-50, 58, 62, 63

2.2 Limits Involving Infinity

Limit of Rational Functions in the form $\frac{\infty}{\infty}$

When taking the limit of a **rational function** and both the numerator and denominator is equal to infinity:
Divide the both the numerator and denominator by highest power (degree) of x in the denominator

Ex. Evaluate the following limits:

a. $\lim_{x\to\infty} \frac{3x+4}{6x^2+1}$

b. $\lim_{x\to\infty} \frac{10x^2+7x-9}{8x^2+11x-2}$

c. $\lim\limits_{x \to \infty} \frac{2x^2+3x-5}{7x-10}$

Horizontal Asymptote of Rational Functions

In previous math classes, you may have come across a method to determine horizontal asymptotes following the steps below:

For a rational function $f(x) = \frac{ax^m+\cdots}{bx^n+\cdots}$,

This method is effective for the most part; but most students cannot explain why it is true.

A better way to find horizontal asymptotes of rational functions can be done using limits! Horizontal asymptotes are pseudo-asymptotes; they can be crossed and are used to predict the end behaviour of a function.

Using Limits to Determine Horizontal Asymptotes

Horizontal asymptotes can be determined by evaluating the limit of a rational function $f(x)$ as x goes to $-\infty$ or ∞.

Typically, you need only evaluate $x \to \infty$ or $x \to -\infty$, unless your $f(x)$ involves absolute values or is a piece-wise function.

Ex. Find the horizontal asymptote of the rational function $f(x) = \frac{2x^2+1}{x^2+x+1}$. Use limits to justify your answer.

Ex. Find the horizontal asymptote of the rational function $f(x) = \frac{5x+11}{3x^2+7}$. Use limits to justify your answer.

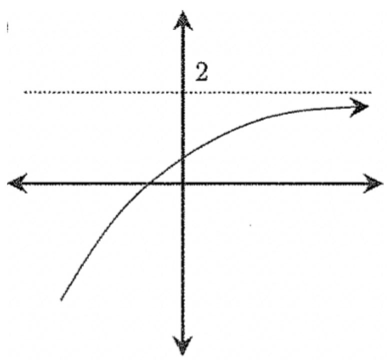
Horizontal Asymptote (In general)

If the line $y = L$ is a horizontal asymptote of the graph of $y = f(x)$, then

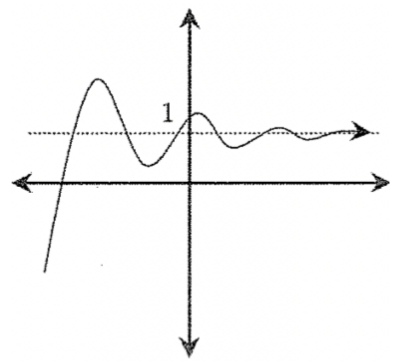
$$\lim_{x \to -\infty} y = L \qquad \text{or} \qquad \lim_{x \to \infty} y = L$$

Ex. Find the equation of the horizontal asymptote for each curve. Use limits to justify your answers.

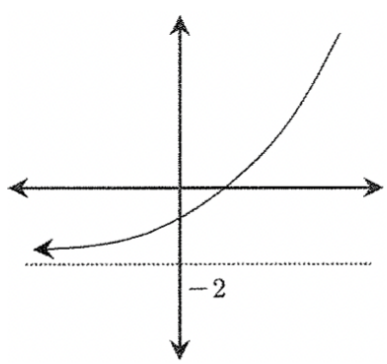
a.



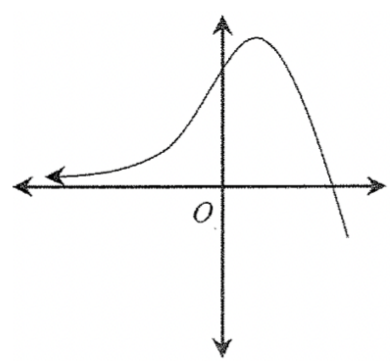
b.



c.



d.

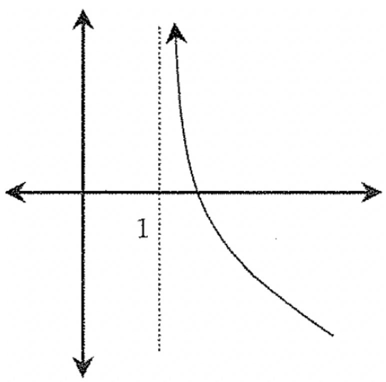


Vertical Asymptote

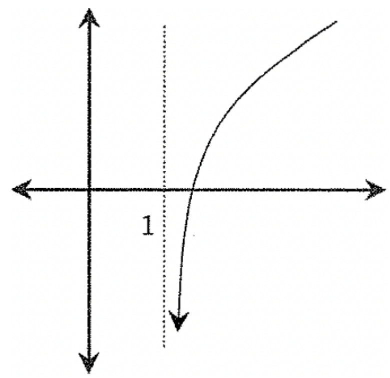
If the line $x = a$ is a vertical asymptote of the graph of $y = f(x)$, then

Ex. Find the equation of the vertical asymptote for each curve. Use limits to justify your answers.

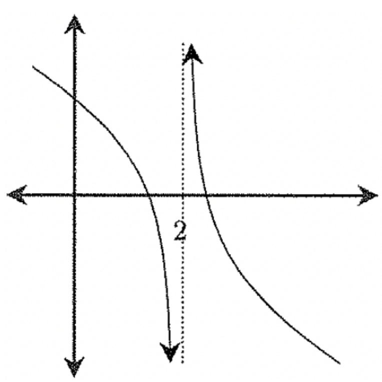
a.



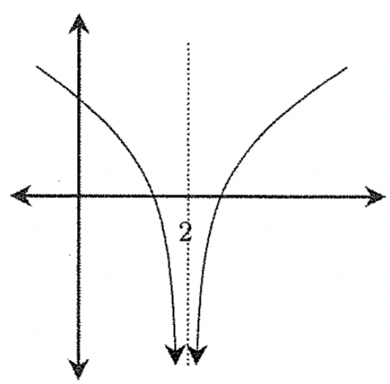
b.



c.



d.

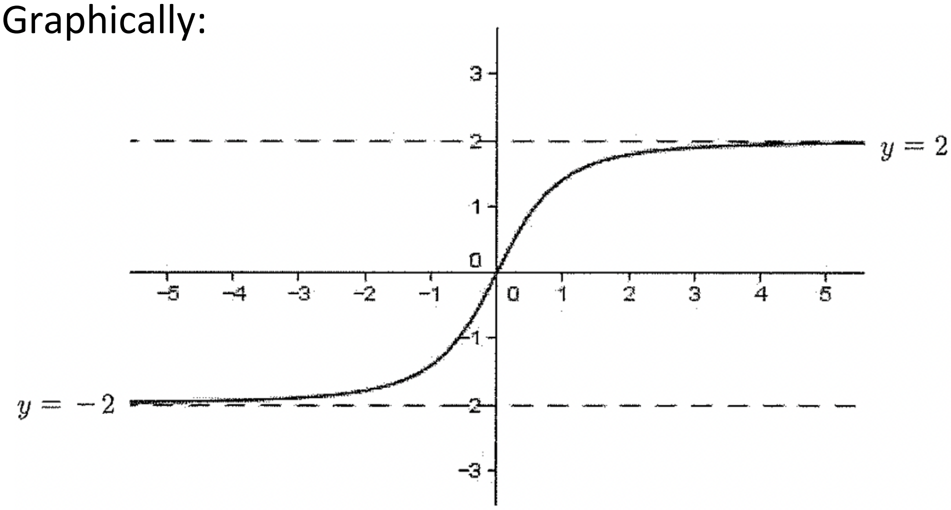


Horizontal Asymptotes Involving Roots/Absolute Value

Recall: $\sqrt{x^2} = |x|$ $|x| = \begin{cases} -x, & x < 0 \\ x, & x \geq 0 \end{cases}$

Ex. Find the equation(s) of the horizontal asymptote(s) of $y = \frac{2x}{\sqrt{x^2+1}}$.

Graphically:



Ex. Find the equation(s) of the horizontal asymptote(s) of $y = \frac{x-4x^2}{\sqrt{7x^4+x^2}}$.

2.2 Homework:
Limits Involving Infinity Worksheet
Stewart:
pg 140 # 3, 5, 7, 14-29, 31, 32, 41-45

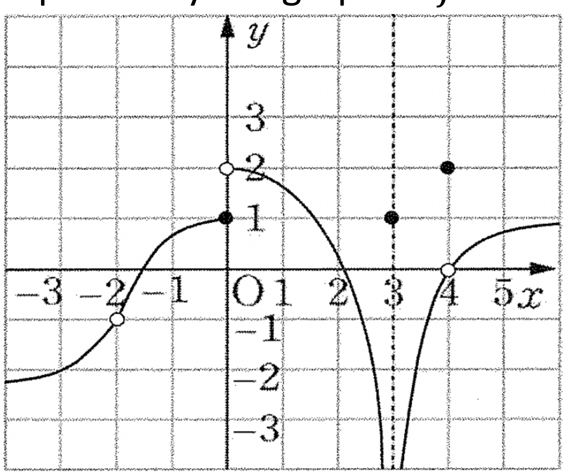
2.3 Continuity

Three-part Definition of Continuity

A function f is **continuous** at $x = a$ if:

All 3 conditions must be met for a function f to be continuous at $x = a$.

Ex. Explain why the graph of f is discontinuous at $x = -2, 0, 3, 4$



At $x = -2$,

At $x = 0$,

At $x = 3$,

At $x = 4$,

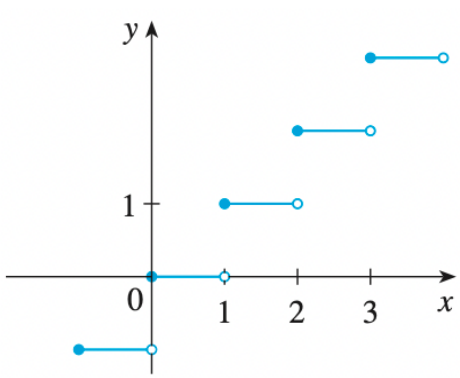
Ex. For the function $f(x) = \frac{x+1}{x^2-x-2}$, answer the following questions.

a. Simplify the equation if possible and sketch the graph of $f(x)$

b. Determine all values of x where f is discontinuous.

c. Find the limit of f at each of the discontinuous points.

Ex. Identify where the function is discontinuous and the type of discontinuity.
$f(x) = \lfloor x \rfloor$ (the greatest integer function)



Ex. Let f be defined by $f(x) = \begin{cases} x^2 + 2x + 3\ , x < 0 \\ \sqrt{x+1} + 2\ \ , x \geq 0 \end{cases}$. Use the **three-part definition of continuity** to prove f is continuous at $x = 0$.

a. $f(0) = \sqrt{0+1} + 2$

b. Left-side limit

Right-side limit:

c. Since $\lim_{x \to 0} f(x) = f(0) = 3$

Functions with Continuous Domain

The following types of functions are continuous in every number in their domains:

Polynomials Functions
Trigonometric Functions
Exponential Functions
Radical functions

Rational functions
Inverse Trigonometric Functions
Logarithmic Functions

Ex. Where is the function $f(x) = \frac{\ln x + \tan^{-1} x}{x^2 - 1}$ continuous?

Ex. Find the value of k that makes $f(x)$ continuous everywhere.

$$f(x) = \begin{cases} \frac{x^2 - 9}{x + 3} & , x \neq -3 \\ k & , x = -3 \end{cases}$$

Ex. $f(x) = \begin{cases} k^3 + x, x < 3 \\ \frac{16}{k^2 - x}, x \geq 3 \end{cases}$

Let f be the function defined above, where k is a positive constant. For what value of k, if any, is f continuous?

2.3 Homework:

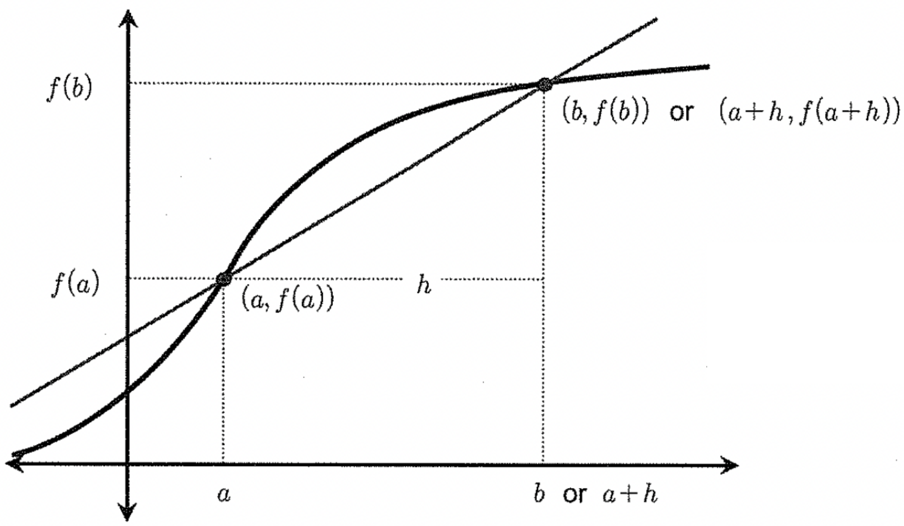
Continuity Worksheet

Stewart:

pg 127 # 3, 4, 12, 14, 16-22, 35, 39, 42, 45, 46

2.4 Average Rate of Change and Instantaneous Rate of Change

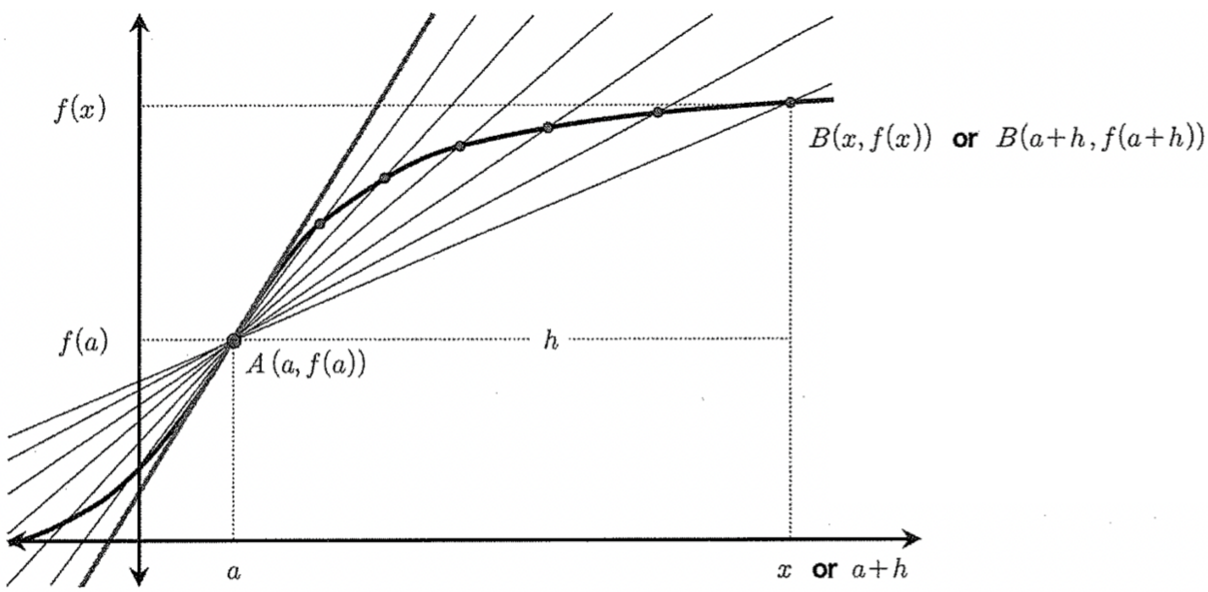
The Average Rate of Change of f with respect to x over $a \leq x \leq b$



Ex. Find the average rate of change for $f(x) = x^2 + 1$, with respect to x over the interval $[-1, 2]$.

The instantaneous Rate of Change of f with respect to x at $x = a$.

The instantaneous rate of change at $x = a$ for $f(x)$ is given by:



The **tangent line** is the line that touches a curve at just one point (**tangent point**). The tangent line is the line that passes through two infinitely close points on a curve.

As point B approaches point A,

Finding the Slope Using $f'(a) = \lim_{x \to a} \frac{f(x)-f(a)}{x-a}$

Ex. Find the equation of the tangent line to the curve $f(x) = x^2 + 1$ at $(2, 5)$.

Ex. Find the equation of the tangent line to the curve $f(x) = 2\sqrt{x}$ at $(4, 4)$.

Finding the Slope Using $f'(a) = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$

Ex. Find the equation of the tangent line to the curve $f(x) = x^2 + 1$ at $(2, 5)$.

Ex. Find the equation of the tangent line to the curve $f(x) = 2\sqrt{x}$ at $(4, 4)$.

Ex. Find the slopes of the tangent line to the graph of the function $f(x) = \sqrt{x}$ at the points $(1,1)$, $(4,2)$, and $(9,3)$.

Velocity Example

Velocity is the rate that displacement changes over time:

Displacement at t seconds:

$v_1 = 0, a = 9.8$:

Function Notation:

Ex. Suppose a ball is dropped from a cliff that is 500 m above ground level.

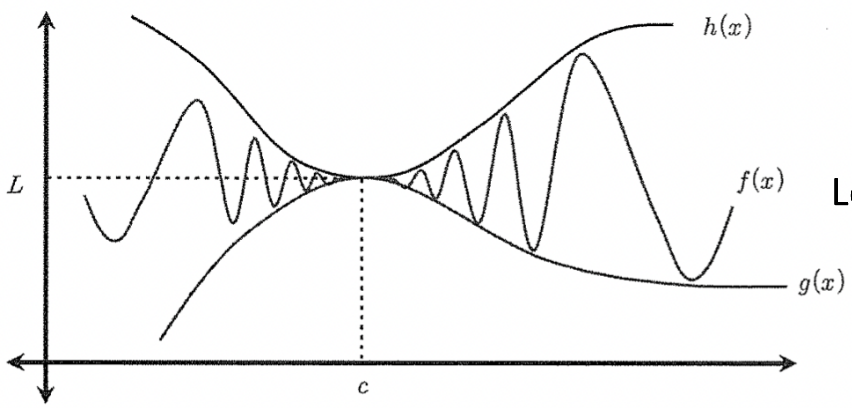
a. Determine the velocity of the ball after 6 seconds.

b. Determine the velocity when the ball hits the ground.
Cliff is 500 m high, so when the ball hits the ground, $d = 500$
First, determine the time it takes to hit the ground.

2.4 Homework:
Avg and Instantaneous Rate of Change Worksheet
Stewart:
pg 150 # 5-8, 10, 13, 16, 18, 20, 25, 27-32, 33, 34, 38-40, 46, 53* fun :)

2.5 Squeeze Theorem

Squeeze Theorem



Let $g(x) \leq f(x) \leq h(x)$.

Ex. Prove that $\lim_{x \to 0} x^2 \cos\left(\frac{1}{x}\right) = 0$

Ex. Given $\frac{1}{x^2+1} \leq f(x) \leq 1$ for all x, find $\lim_{x \to 0} f(x)$

Ex. Prove that $\lim_{x \to 0} x^4 \sin^2\left(\frac{1}{x}\right) = 0$

2.5 Homework:

Squeeze Theorem Worksheet

Stewart:

Pg 107 # 36-40

Pg 142 # 57

2.6 Limits of Trigonometric Functions

Limits of Trigonometric Functions at asymptotes

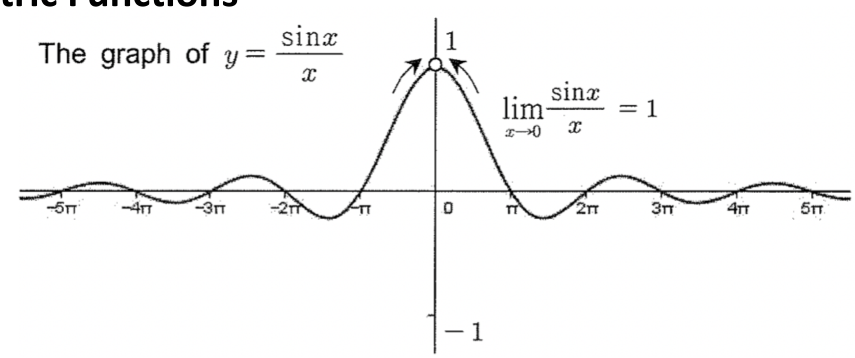
For $y = \tan x$, asymptotes at $x = \frac{\pi}{2} + \pi n, n \in \mathbb{Z}$

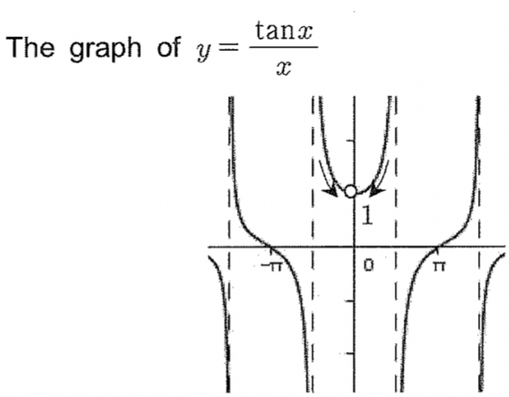
For $y = \cot x$, asymptotes at $x = \pi n, n \in \mathbb{Z}$

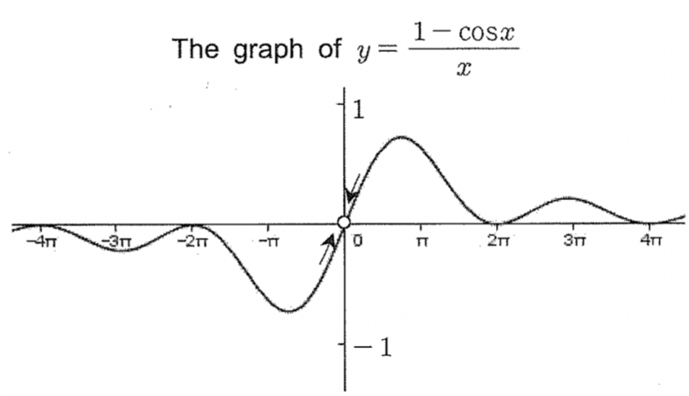
For $y = \sec x$, asymptotes at $x = \frac{\pi}{2} + \pi n, n \in \mathbb{Z}$

For $y = \csc x$, asymptotes at $x = \pi n, n \in \mathbb{Z}$

Limits of Trigonometric Functions







Ex. Evaluate the following limits.

a. $\lim_{x \to 0} \frac{\sin 3x}{3x}$

b. $\lim_{x \to 0} \frac{\tan 2x}{2x}$

c. $\lim_{x \to 0} \frac{\tan(\sin x)}{\sin x}$

d. $\lim_{x \to \infty} x \sin\left(\frac{1}{x}\right)$

e. $\lim_{x \to 0} \frac{\sin x}{3x}$

f. $\lim_{x \to 0} \frac{\tan 2x}{\frac{2}{3}x}$

g. $\lim_{t \to 0} \frac{\sin 5t}{t}$

h. $\lim_{\theta \to 0} \frac{\tan 2\theta}{3\theta}$

i. $\lim_{x \to 0} \frac{\sin 5x}{\tan 2x}$

j. $\lim_{x \to 0} \frac{\sin(\tan x)}{2x}$

2.6 Homework:

Limits of Trigonometric Functions Worksheet

Stewart:

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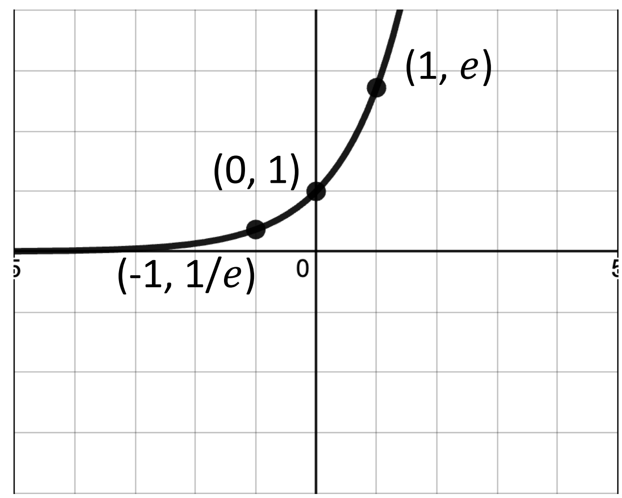
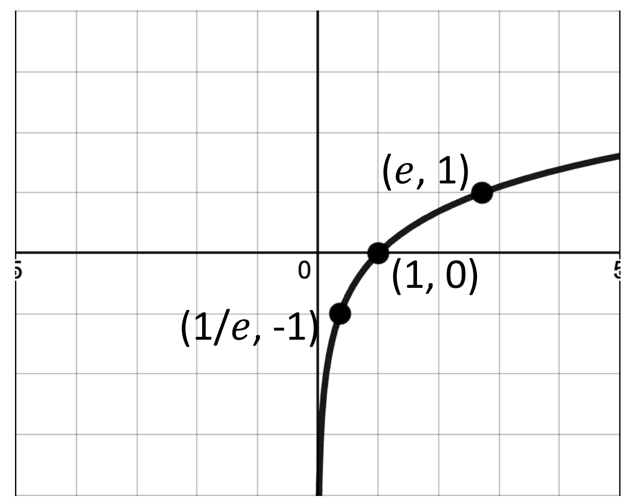
2.7 Limits of Exponential and Logarithmic Functions

$y = e^x$



$y = \ln x$



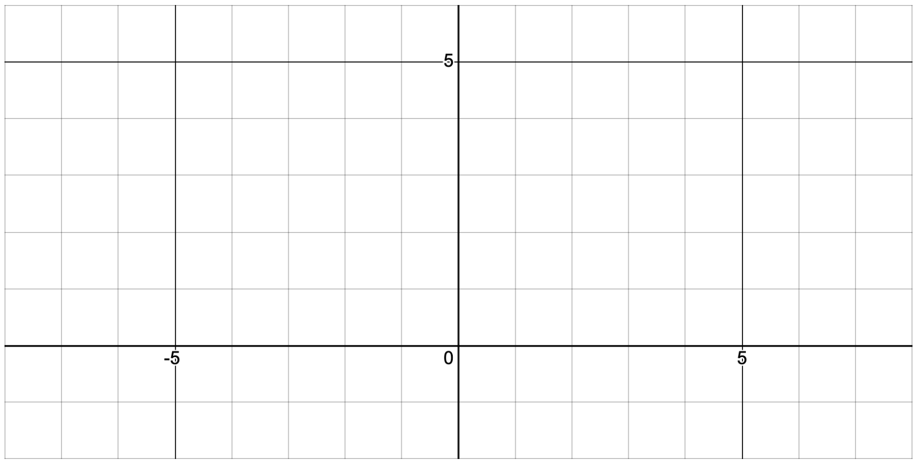
Ex. Evaluate the following limits.

a. $\lim\limits_{x \to \infty} 3^x$

b. $\lim\limits_{x \to \infty} 3^{-x}$

c. $\lim\limits_{x \to \left(\frac{\pi}{2}\right)^-} \left(\frac{1}{3}\right)^{\tan x}$

Definition of Euler's Number, e



Ex. Evaluate the following limits.

a. $$\lim_{x\to\pm\infty}\left(1+\frac{2}{x}\right)^{\frac{x}{2}} = \lim_{x\to\pm\infty}\left(1+\frac{5}{x}\right)^{\frac{x}{5}} = \lim_{x\to\pm\infty}\left(1-\frac{3}{2x}\right)^{-\frac{2x}{3}}$$

b. $$\lim_{x\to 0}(1+2x)^{\frac{1}{2x}} = \lim_{x\to 0}\left(1+\frac{x}{3}\right)^{\frac{3}{x}} = \lim_{x\to 0}\left(1-\frac{x}{3}\right)^{-\frac{3}{x}}$$

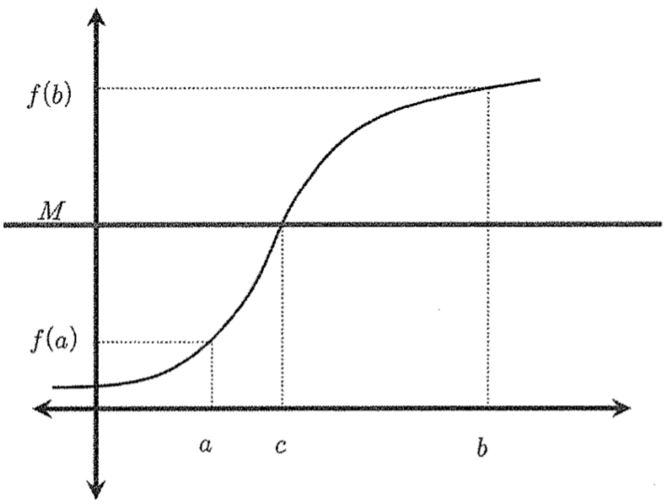
c. $$\lim_{x\to\infty}\left(1+\frac{4}{x}\right)^{x}$$

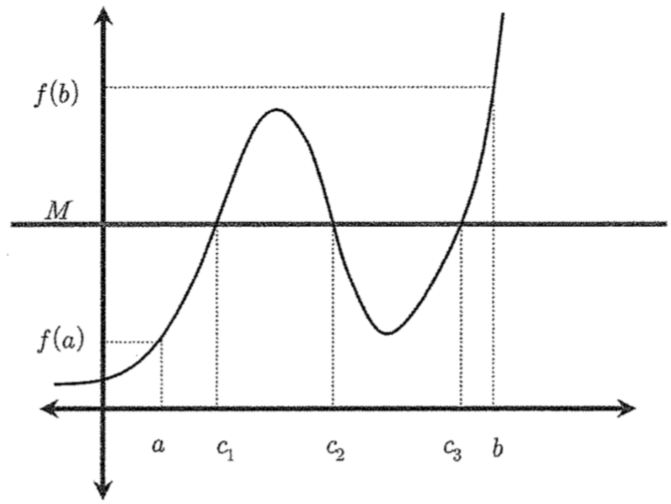
d. $\lim_{x \to \infty} \left(1 - \frac{3}{x}\right)^x$

2.7 Homework:

Limits of Exponential and Logarithmic Functions Worksheet

2.8 Intermediate Value Theorem (IVT)





Ex. Show that the equation $x^3 - 2x - 2 = 0$ has a solution on the interval $[1, 2]$.

Ex. Show that there is a solution of $\sqrt[3]{x} + x = 1$ in the interval $[0, 8]$.

2.8 Homework:

Intermediate Value Theorem (IVT) Worksheet #All

Stewart:

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