Chapter 5 – Exponents and Logarithms

5.1 Exponents

Recall the exponent laws:

1. $b^0 = 1 \quad , \quad b \neq 0$
2. $b^1 = b$
3. $b^m \times b^n = b^{m+n}$
4. $\frac{b^m}{b^n} = b^m \div b^n = b^{m-n}$
5. $(b^m)^n = b^{mn}$
6. $b^{-1} = \frac{1}{b}$

Remember: $f^{-1}(x) \neq \frac{1}{f(x)} = [f(x)]^{-1}$

7. $\left(\frac{a}{b}\right)^{-m} = \left(\frac{b}{a}\right)^m = \frac{b^m}{a^m}$
8. $(ab)^m = a^m b^m$
9. $b^m = b^n$ if and only if $m = n$

Ex. Simplify $32^x \cdot 16^{3x}$

Re-write 32 and 16 as powers of 2

$= 2^{5 \cdot x} \cdot 2^{4 \cdot 3x}$

$= 2^{5x} \cdot 2^{12x}$

$= 2^{17x}$

Ex. Simplify $\frac{8^{2x}}{4^3}$

$= \frac{2^{6x}}{2^6}$

$= 2^{6x-6}$

Ex. Simplify $\frac{27^{x+3}}{81^{x-3}}$

Re-write 27 and 81 as powers of 3

$$= \frac{3^{3(x+3)}}{3^{4(x-3)}}$$

$$= \frac{3^{3x+9}}{3^{4x-12}}$$

$$= 3^{-x+21}$$

Ex. Simplify $\frac{32^x \cdot \left(\frac{1}{8}\right)^x}{\left(\frac{1}{64}\right)^x \cdot 16^{2x}}$

$$= \frac{2^{5x} \cdot 2^{-3x}}{2^{-6x} \cdot 2^{8x}}$$

$$= \frac{2^{2x}}{2^{2x}}$$

$$= 1$$

Ex. Solve $64^{6x+3} = 8^{4x+2}$

* re-write each side as a common "power"

$8^{2(6x+3)} = 8^{4x+2}$ $\qquad$ $2^{6(6x+3)} = 2^{3(4x+2)}$

$8^{12x+6} = 8^{4x+2}$ $\qquad$ $2^{36x+18} = 2^{12x+6}$

$\therefore$ $12x + 6 = 4x + 2$ $\qquad$ or $\qquad$ $36x + 18 = 12x + 6$

$8x = -4$ $\qquad$ $24x = -12$

$x = -\frac{1}{2}$ $\qquad$ $x = -\frac{1}{2}$

Compound interest problems

Compound interest can be found using the following formula:

$$A = P\left(1 + \frac{r}{n}\right)^{tn}$$

where,

$A -$ final amount, includes principle and interests earned, $A = P + I$

$P -$ initial amount (Principle)

$r -$ yearly interest rate (decimal form)

$n -$ number of compounding periods in a year

$t -$ time in years (term)

Compounding Periods

$n = 1$	annually	$n = 2$	semi-annually
$n = 4$	quarterly	$n = 12$	monthly
$n = 52$	weekly	$n = 365$	daily

Ex. \$10,000 is invested in a stock that yields a return of 10% annually. Determine the value of the investment after 12 years. Approximate answer to the nearest cent.

Identify all known and unknown information.

$A = ?$ $\quad P = 10000 \quad r = 0.1 \quad t = 12 \quad n = 1$

$$A = P\left(1 + \frac{r}{n}\right)^{tn}$$

$$= 10000(1 + 0.1)^{12} = 31384.28$$

The investment will have a value of $\$31{,}384.28$ in 12 years.

Ex. Jacob has \$5000 to invest. He plans to invest it over 10 years, yielding 8% per annum, and compounded semi-annually. Determine the amount of interest Jacob earns at the end of 10 years.

$A = ?$ $P = 5000$ $r = 0.08$ $n = 2$ $t = 10$

$$A = 5000\left(1 + \frac{0.08}{2}\right)^{2(10)}$$

$$A = \$10{,}955.62$$

$I = A - P$ or $A = P + I$ ($I =$ interest earned)

$$I = 10955.62 - 5000$$

$$I = 5955.62$$

Jacob earns \$5955.62 in interest over the 10 years.

Ex. Which is the better option?

Investment A:

$P = \$5000$

$r = 4\% = 0.04$

$t = 10$

$n = 12$

A:

$$A = 5000\left(1 + \frac{0.04}{12}\right)^{12(10)}$$

$$= 7454.16$$

Investment B:

$P = \$5000$

$r = 4.05\% = 0.0405$

$t = 10$

$n = 1$

B:

$$A = 5000\left(1 + \frac{0.0405}{1}\right)^{10}$$

$$= 7436.88$$

∴ Investment A is the better deal

Growth and Decay

For growth and decay questions, the future population or amount can be found using the formula:

$$A = A_o(x)^{\frac{t}{T}}$$

A final amount

A_o starting amount

x growth / decay number

t the time that has elapsed

T the period of growth / decay

Growth and Decay Number, x

When:

$x = 2 \rightarrow$ doubling $x = 3 \rightarrow$ tripling

$x = \frac{1}{2} \rightarrow$ decrease by 50% or half-life

$x = \frac{1}{4} \rightarrow$ decrease by 75%

Ex. A population of wild dogs is currently at 30. The population will **triple** in 50 days. Determine the population after 21 days.

$A = 30(3)^{\frac{21}{50}}$

$= 47.5896$

≈ 47

The population of wild dogs will be 47 after 21 days.

Ex. The **half-life** of plutonium-123 is about 15000 years. How much of a given sample will remain after 4000 years?

$A_0 = 1$ or 100% $\qquad x = \frac{1}{2} \qquad t = 4000 \qquad T = 15000$

$$A = 1\left(\frac{1}{2}\right)^{\frac{4000}{15000}}$$

$$A = 0.831$$

83.1% of the sample will remain after 4000 years.

Ex. The number of tapeworms increases by 40% every 3 days. If the population is 150 tapeworms after 9 days, what was the initial population?

$A = 150 \qquad A_0 = ? \qquad x = 1.4 \qquad t = 9 \qquad T = 3$

$$150 = A_0(1.4)^{\frac{9}{3}}$$

$$A_0 = \frac{150}{(1.4)^3} = 54.664$$

≈ 55 tapeworms

Or

Alternatively, 9 days ago could be interpreted as $t = -9$

$$A = 150(1.4)^{\frac{-9}{3}}$$

$$A = 54.664$$

≈ 55 tapeworms

The original population had 55 tape worms.

Ex. If an earthquake in Tokyo had an amplitude 10000 times larger than an earthquake that measured 3.2 on the Richter scale, what would the Tokyo earthquake measure?

The magnitude of an earthquake is the measure of its strength
The magnitude of a 3.2 earthquake is $10^{3.2}$

If another earthquake is 10000 (or 10^4) times greater in magnitude than the 3.2 earthquake, then the other earthquake would have a magnitude of:

$$10^{3.2} \times 10^4 = 10^{7.2}$$

∴ the reading on the Richter scale would be 7.2

Constant Growth Equation

For questions where the growth number is not known, use the constant growth equation:

$$\boxed{A = A_o e^{kt}}$$

A final amount

A_o starting amount

e natural constant ≈ 2.71828 (Euler's number)

k proportional constant

t time elapsed

Unfortunately, using this equation requires logarithms (which will be covered later in the unit.

A demonstration of this equation will be shown for the half-life problem that was previous solved.

Ex. The **half-life** of plutonium-123 is about 15000 years. How much of a given sample will remain after 4000 years?

$A_0 = 1$ $\qquad$ $k = ?$

Half-life is 15000 years, use to solve for k

$$\mathrm{A} = \mathrm{A}_0 e^{kt}$$

$$\frac{1}{2} = 1e^{k(15000)}$$

$$n = b^x \quad \rightarrow \quad \log_b n = x$$

$$\frac{1}{2} = e^{k(15000)} \quad \rightarrow \quad \log_e\left(\frac{1}{2}\right) = 15000k$$

Recall: $\log_e(0.5) = \ln(0.5)$

$\therefore \quad 15000k = \ln(0.5)$

$$k = \frac{\ln\left(\frac{1}{2}\right)}{15000} \quad \text{or} \quad k = \frac{1}{15000}\ln\left(\frac{1}{2}\right)$$

$$A = 1e^{\frac{\ln\left(\frac{1}{2}\right)}{15000}t} \quad \text{or} \quad A = 1e^{\frac{1}{15000}\ln\left(\frac{1}{2}\right)}$$

Sub in $t = 4000$

$$A = 1e^{\frac{\ln(0.5)}{15000}\times(4000)}$$

$A = 0.831$ or 83.1%

83.1% of the sample will remain after 4000 years.

5.1 Homework (Part I)

1, 2, 9 acde

Exponential Graphs

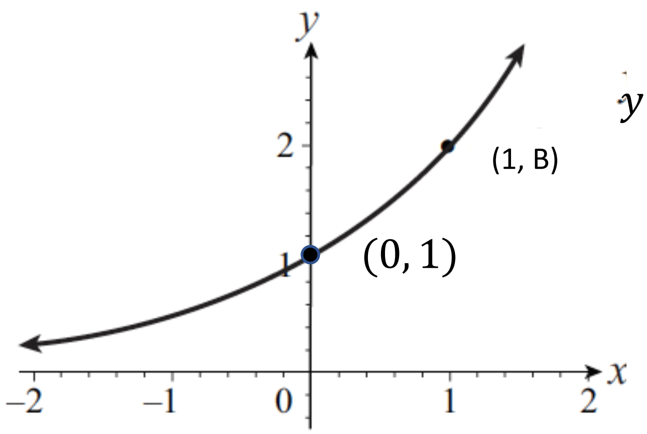
$$y = aB^{b(x-c)} + d$$

In this chapter we typically will not have b values that is not equal 1 (for graphing)
∴ no horizontal compressions or expansions, possible horizontal reflection

Exponential graphs can only have base value, B, that is greater than 0 and not equal to 1 $\quad B > 0$ and $B \neq 1$

For exponential functions, where B is the base:

When $B > 1$, the graph has the following general shape:

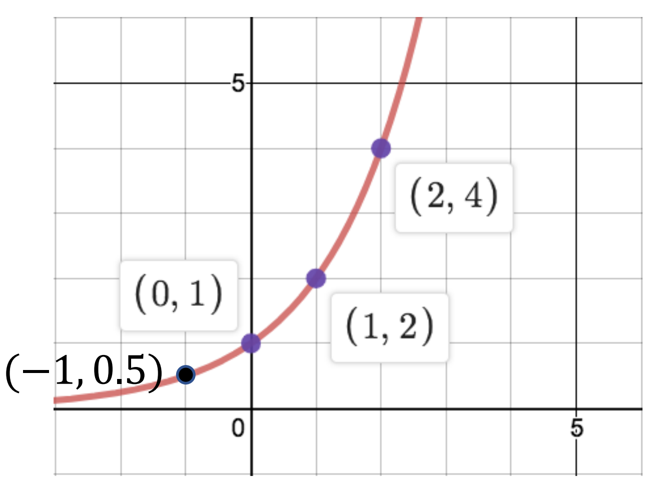


$y = B^x, B > 1$

Ex. The graph of $y = 2^x$

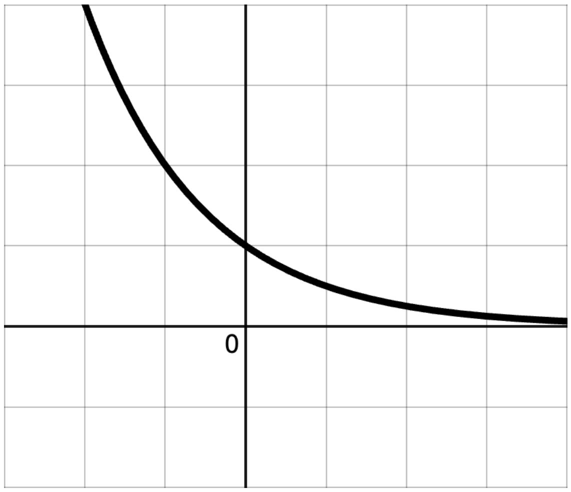
In this example, the base is greater than 1

x	-1	0	1	2
y	0.5	1	2	4



HA: $y = 0$

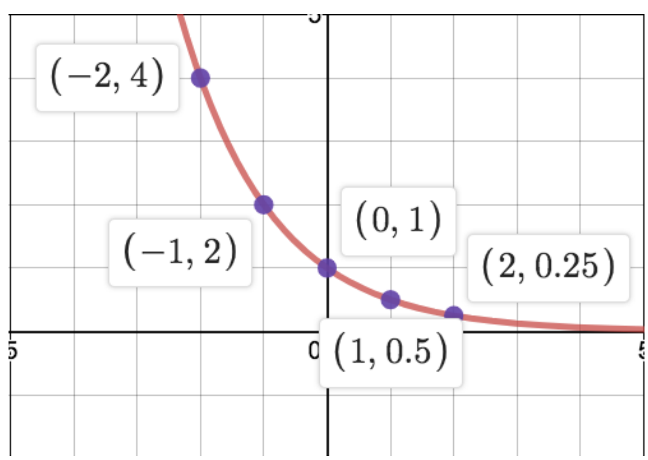
When $0 < B < 1$, the graph has the following general shape:



Ex. The graph of $y = \left(\frac{1}{2}\right)^x$.

In this scenario, the base is between 0 and 1.

$y = 2^{-x}$, this is a reflection over the y-axis (horizontal reflection) of $y = 2^x$.



HA: $y = 0$

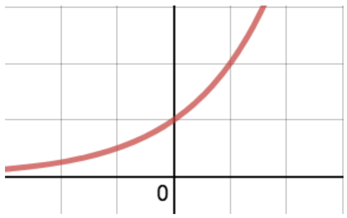
From the two graphs above, we can conclude that:

When $B > 1$, the exponential graph is increasing.

While $0 < B < 1$, the exponential graph is decreasing.

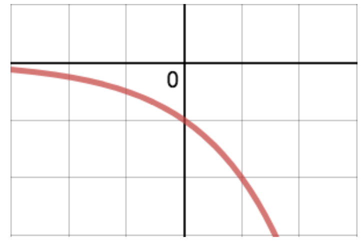
Ex. Is the exponential graph, $y = aB^x$, increasing or decreasing?

a. $B > 1, a > 0$



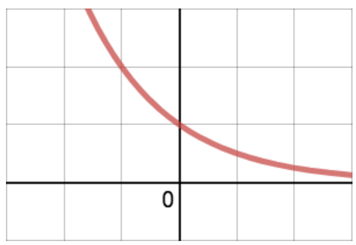
> the graph is increasing

b. $B > 1, a < 0$



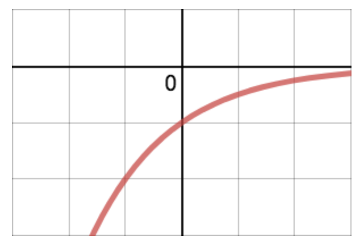
> the graph is decreasing

c. $0 < B < 1, a > 0$



> the graph is decreasing

d. $0 < B < 1, a < 0$



> the graph is increasing

Horizontal Asymptote

For the general exponential function $y = aB^{x-c} + d$:
the graph has a horizontal asymptote at $\boldsymbol{y = d}$

Ex. Determine the horizontal asymptote of the following exponential functions.

a. $y = 2^x$

has a HA: $y = 0$

b. $y = -\left(\frac{1}{2}\right)^x - 3$

has a HA: $y = -3$

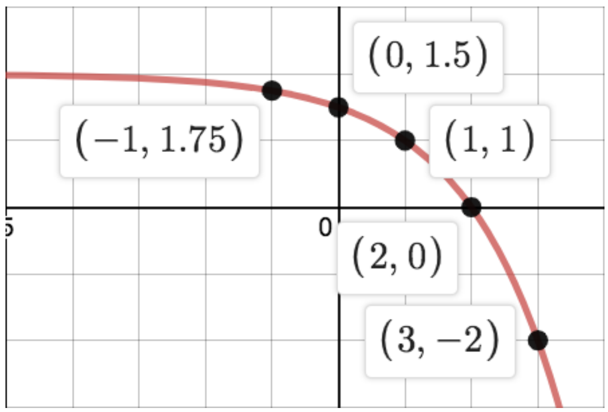
c. $y = e^{x-2} + 1$

has a HA: $y = 1$

Ex. Using a table of values, graph $y = -2^{x-1} + 2$

x	-1	0	1	2	3
y	1.75	1.5	1	0	-2

Note: We picked 1 as the middle value for x, because of the horizontal translation



Another example of changing a number to be a common base and combining with another term.

Ex. Simplify $y = 8 \cdot 2^{x+1}$

$$y = 2^3 \cdot 2^{x+1}$$

$$y = 2^{x+4}$$

In this situation, a vertical expansion by a factor 8 is equivalent to a horizontal translation of -3

Ex. Simplify $y = \frac{1}{125}(5)^{x+2}$

$$y = 5^{-3} \cdot (5)^{x+2}$$

$$y = 5^{x-1}$$

5.1 Homework (Part II)
3 – 8, 9f

5.2 Logarithmic Functions and Their Graphs

Logarithmic Functions

1. They are one-to-one functions
2. The inverses of Logarithms are Exponential Functions
3. Logarithmic functions are reflections of exponential functions over the $y = x$ line
4. The domain of a log function, is the range of its corresponding exponential function
5. The range of a log function, is the domain of its corresponding exponential function

Exponential form and Logarithmic form

Ex. Find the inverse of $y = b^x$

$f^{-1}(x)$: $x = b^y$ (exponential form)

This is the process of converting between exponential and logarithmic form

$y = \log_b x$ (logarithmic form)

Examples of converting between the two forms:

Exponential form	Logarithmic Form
$y = b^x$	$x = \log_b y$
$25 = 5^2$	$2 = \log_5 25$
$2d + 1 = (6f)^{g-1}$	$g - 1 = \log_{6f}(2d + 1)$

Ex. Change to the other form

a. $\log_5\left(\frac{1}{125}\right) = -3$

$5^{-3} = \frac{1}{125}$

b. $\log_2 8 = 3$

$2^3 = 8$

c. $81 = 3^4$

$\log_3(81) = 4$

d. $2^{-6} = \frac{1}{64}$

$\log_2\left(\frac{1}{64}\right) = -6$

Fact: $\log x = \log_{10} x$ by default $\log x$ is base 10

$\therefore \log_{10} 1000 = \log 1000$ $\log_{10} 5 = \log 5$

Solving Equations Using Logarithms

Ex. Solve $4^x = 28$

To isolate a variable that is in the exponent, we need to change from exponential form to logarithmic form

$x = \log_4 28$ (this is the exact value)

Better answer is $1 + \frac{1}{2}\log_2 7$, which we will see how to get in section 5.3.

If an approximate answer is required, depending on the functionality of your calculator, you may need to do one of the following:

$x = \log_4 28 \approx 2.40$

If your calculator does not have this function, Change of Base Rule is required:

$$\log_b n = \frac{\log n}{\log b}$$

$x = \log_4 28$

$= \frac{\log 28}{\log 4}$

≈ 2.40

Ex. Solve for x. $5 = 2^x$

Convert to log form

$x = \log_2 5$ or $x \approx 2.32$

Characteristics of Logarithmic Functions

Since logarithmic functions are the inverse of exponential functions, the domain and ranges of each function are the reverse of the other:

	$y = B^x$		$y = \log_B x$
D:	$\{x \in \mathbb{R}\}$	D:	$\{x \in \mathbb{R} \mid x > 0\}$
R:	$\{y \in \mathbb{R} \mid y > 0\}$	R:	$\{y \in \mathbb{R}\}$
HA:	$y = 0$	VA:	$x = 0$

Restrictions on Logarithmic Functions

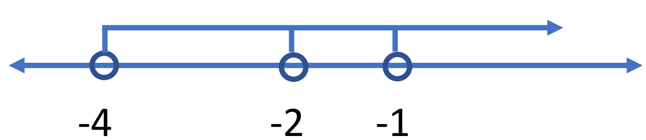
The restrictions on the base of a logarithmic function, are the same as an exponential function.

In addition, you cannot find the log of a negative number: $\log(-5) = DNE$

$\boldsymbol{y = \log_B N}$ has the following restrictions: $\boldsymbol{N > 0,\quad B > 0,\quad B \neq 1}$

Ex. Determine the restrictions on $y = \log_{x+2}(x+4)$.

$x + 4 > 0$	$x + 2 > 0$	$x + 2 \neq 1$
$x > -4$	$x > -2$	$x \neq -1$



Combining the restrictions, the result is:

$\therefore \quad x > -2, x \neq -1$

Inverse of exponential and logarithmic functions.

Ex. Find the inverse of $y = 2^{x+3} - 1$

$f^{-1}(x)$: $x = 2^{y+3} - 1$

$x + 1 = 2^{y+3}$

$y + 3 = \log_2(x + 1)$

$y = \log_2(x + 1) - 3$

Ex. Find the inverse of $y = \log_3(x - 2) + 4$

$f^{-1}(x)$: $x = \log_3(y - 2) + 4$

$x - 4 = \log_3(y - 2)$

$y - 2 = 3^{x-4}$

$y = 3^{x-4} + 2$

Ex. The graph $y = b^x$ has a coordinate at (e, f). What coordinate must be on $y = 2\log_b(-x + 3) - 5$?

$$y = 2\log_b\big(-(x - 3)\big) - 5$$

Mapping:

$(x, y) \rightarrow (-y + 3, 2x - 5)$

$(e, f) \rightarrow (-f + 3, 2e - 5)$

5.2 PI Homework

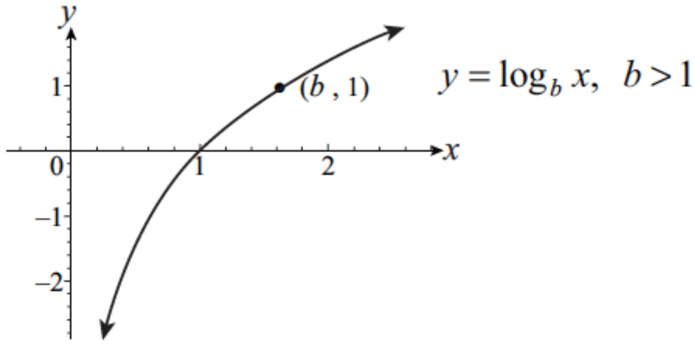
1-5, 8 – 10, 13

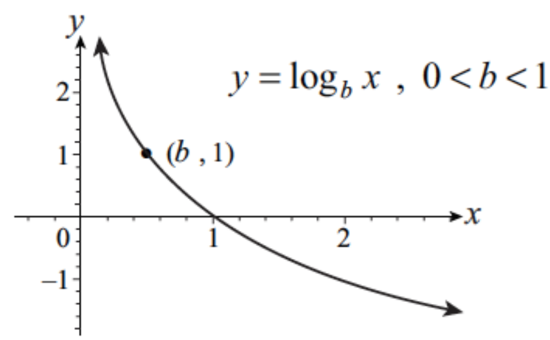
Graphs of Logarithmic Functions

Logarithmic Function with transformations $f(x) = a\log_B\big(b(x-c)\big) + d$

- a vertical exp / comp and reflection over x-axis
- b horizontal exp / comp and reflection over y-axis
- c horizontal translation
- d vertical translation

General shape of logarithmic functions, depending on the value of the base:

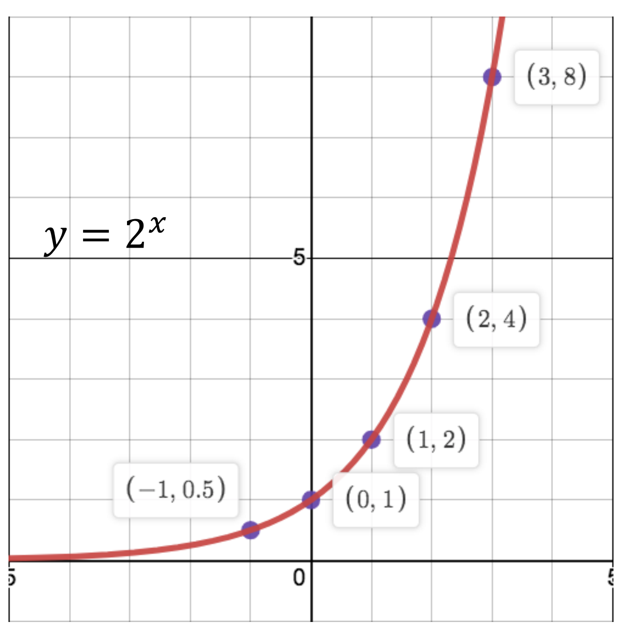


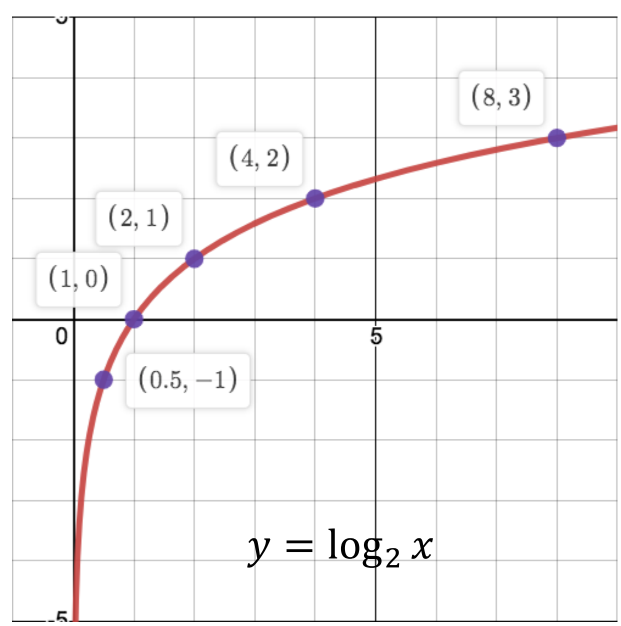


When $B > 1$, the $f(x)$ is increasing

When $0 < B < 1$, $f(x)$ is decreasing

Comparing the graphs of $y = 2^x$ and $y = \log_2 x$

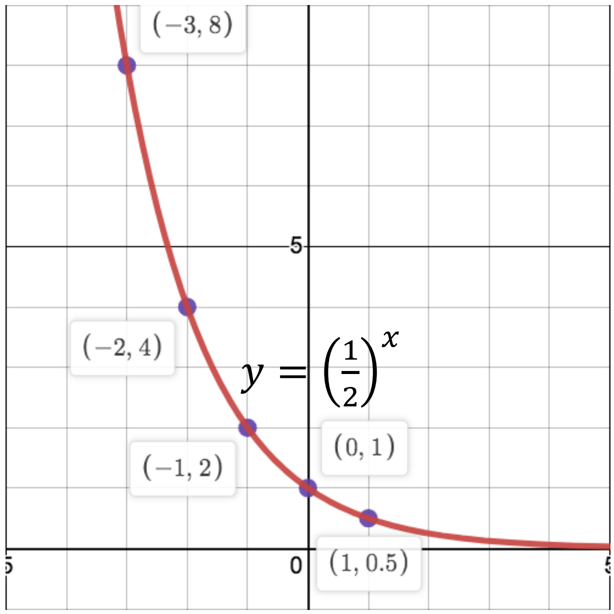


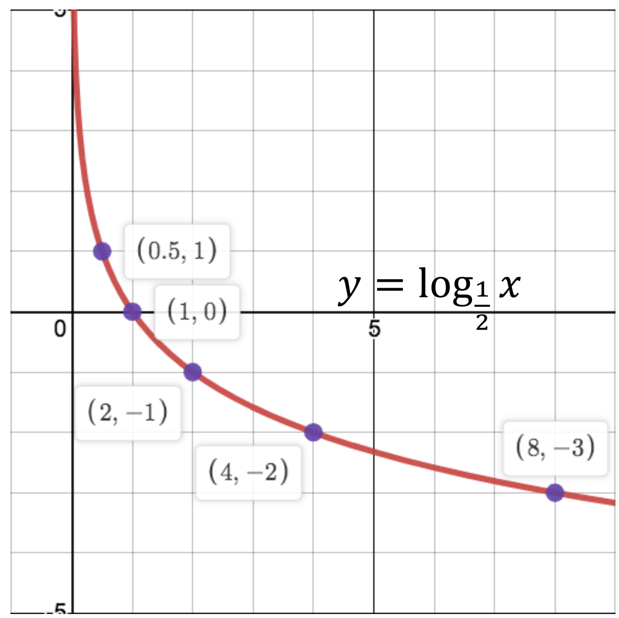


D: $\{x \in \mathbb{R}\}$
R: $\{y \in \mathbb{R} \mid y > 0\}$
HA: $y = 0$

D: $\{x \in \mathbb{R} \mid x > 0\}$
R: $\{y \in \mathbb{R}\}$
VA: $x = 0$

Comparing the graphs of $y = \left(\frac{1}{2}\right)^x$ and $y = \log_{\frac{1}{2}} x$





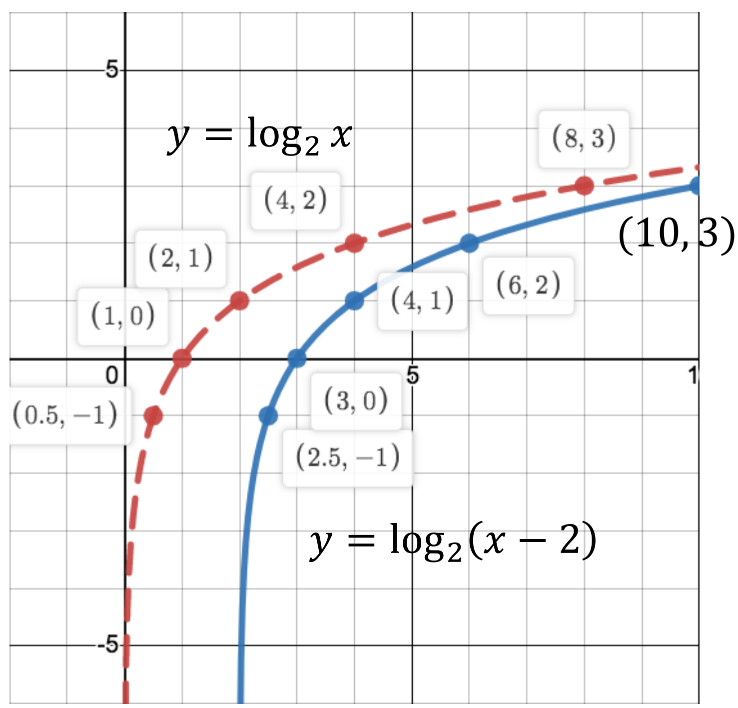
D: $\{x \in \mathbb{R}\}$
R: $\{y \in \mathbb{R} \mid y > 0\}$
HA: $y = 0$

D: $\{x \in \mathbb{R} \mid x > 0\}$
R: $\{y \in \mathbb{R}\}$
VA: $x = 0$

Ex. Graph $y = \log_2(x - 2)$

Mapping: $(x, y) \to (x + 2, y)$

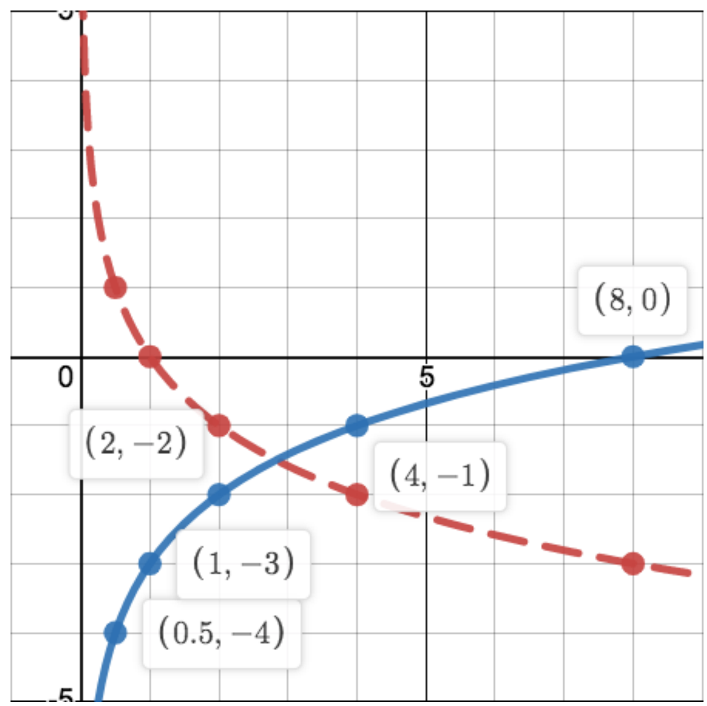
$(0.5, -1) \to (0.5 + 2, -1)$
$= (2.5, -1)$
$(1, 0) \to (1 + 2, 0) = (3, 0)$
$(2, 1) \to (2 + 2, 1) = (4, 1)$
$(4, 2) \to (4 + 2, 2) = (6, 2)$
$(8, 3) \to (8 + 2, 3) = (10, 3)$



Ex. Graph $y = -\log_{\frac{1}{2}} x - 3$

Mapping: $(x, y) \rightarrow (x, -y - 3)$

$(0.5, 1) \rightarrow (0.5, -1 - 3)$
$= (0.5, -4)$
$(1, 0) \rightarrow (1, -0 - 3) = (1, -3)$
$(2, -1) \rightarrow (2, 1 - 3) = (2, -2)$
$(4, -2) \rightarrow (4, 2 - 3) = (4, -1)$
$(8, -3) \rightarrow (8, 3 - 3) = (8, 0)$



5.2 Homework PII
6, 7, 11, 12

5.3 Properties of Logarithms

Product Rule

Recall: $b^m \times b^n = b^{m+n}$

$$\log_b m + \log_b n = \log_b(m \cdot n)$$

Division Rule

Recall: $\frac{b^m}{b^n} = b^{m-n}$

$$\log_b m - \log_b n = \log_b \left(\frac{m}{n}\right)$$

Power Rule

Recall: $(b^m)^n = b^{mn}$

$$\log_b m^n = n \log_b m$$

Change of Base Rule

$$\log_b a = \frac{\log a}{\log b} \qquad \log_b a = \frac{\log_n a}{\log_n b}$$

Note: $\ln x = \log_e x$ (natural log)

Ex. Simplify $\log 12 - \log 4$

> Division Rule

$$= \log \frac{12}{4}$$

$$= \log 3$$

Ex. Simplify $\log_2 8$ using the power rule.

$$= \log_2 2^3$$

Using the power rule

$$= 3 \log_2 2$$

Reminder: $\log_x x = 1$, so $\log_2 2 = 1$

$$= 3$$

5.3 Homework PI: # 1,2

Ex. Write each logarithm in terms of $\log 3$ and $\log 5$

a. $\log 75$

$= \log(25 \times 3)$

Using the product rule

$= \log 25 + \log 3$

Use the power rule on $\log 5^2$

$= \log 5^2 + \log 3$

$= 2\log 5 + \log 3$

b. $\log \frac{125}{81}$

Use division rule

$= \log 125 - \log 81$

Re-write as powers of 5 and 3

$= \log 5^3 - \log 3^4$

Use power rule

$= 3\log 5 - 4\log 3$

Ex. Determine the exact value (without the calculator)

a. $\log_3 \sqrt{243}$

$= \log_3 \sqrt{3^5}$

$= \log_3 3^{\frac{5}{2}}$ Recall: $\sqrt{x^n} = x^{\frac{n}{2}}$ So $\sqrt{3^5} = 3^{\frac{5}{2}}$

$= \frac{5}{2}\log_3 3$

$= \frac{5}{2}$

b. $\log_4 2 + \log_2 32$

$= \log_4 4^{\frac{1}{2}} + \log_2 2^5$ Recall: $2 = \sqrt{4} = 4^{\frac{1}{2}}$

$32 = 2^5$

$= \frac{1}{2}\log_4 4 + 5\log_2 2$

$= \frac{1}{2} + 5$

$= \frac{11}{2}$

Ex. Evaluate $\log_9 27^{1.4}$

Use change of base rule

$= \frac{\log 27^{1.4}}{\log 9}$

Use power rule

$= \frac{1.4 \log 27}{\log 9}$

Re-write 27 as a power of 3, and 9 as a power of 3 in the denominator

$= \frac{1.4 \log 3^3}{\log 3^2}$

Use power rule in the numerator and denominator

$= \frac{3\times 1.4 \log 3}{2\times \log 3}$

$= \frac{4.2 \log 3}{2 \log 3}$

Reduce $\log 3$ over $\log 3$ to equal to 1

$= \frac{4.2}{2}$

$= \frac{21}{10}$

Ex. Simplify $\frac{1}{\log_2 10} + \frac{1}{\log_5 10}$

Use change of base rule

$$= \frac{1}{\frac{\log 10}{\log 2}} + \frac{1}{\frac{\log 10}{\log 5}}$$

$$= \frac{\log 2}{\log 10} + \frac{\log 5}{\log 10}$$

Recall: $\log 2 = \log_{10} 2 = \frac{\log 2}{\log 10}$

$\therefore \frac{\log 2}{\log 10} = \log 2$ and $\frac{\log 5}{\log 10} = \log 5$

$$= \log 2 + \log 5$$

$$= \log 10$$

$$= 1$$

Ex. Simplify $6 \log_9 x - 12 \log_{27} x$. Write as a single logarithm.

Apply Change of Base Rule

$$= \frac{6 \log x}{\log 9} - \frac{12 \log x}{\log 27}$$

Re-write 9 and 27 as power of 3 and then apply the power rule

$$= \frac{6 \log x}{2 \log 3} - \frac{12 \log x}{3 \log 3}$$

$$= \frac{3 \log x}{\log 3} - \frac{4 \log x}{\log 3}$$

$= -\frac{\log x}{\log 3}$ or $= 3 \log_3 x - 4 \log_3 x$

$= -\log_3 x$ $= -\log_3 x$

Ex. Simplify $\log_b x^{\log_x a}$

Use the Exponent Rule

$$= \log_x a \log_b x$$

Use Change of Base Rule, then reduce the fractions.

$$= \frac{\log a}{\log x} \cdot \frac{\log x}{\log b}$$

$$= \frac{\log a}{\log b}$$

$$= \log_b a$$

Ex. Simplify $5^{-3\log_5 2}$

Let the expression equal to y

$$5^{-3\log_5 2} = y$$

Re-write in log form

Base = 5 number = y exponent = $-3\log_5 2$

$$\log_5 y = -3\log_5 2$$

Isolate y

$$\log_5 y = \log_5 2^{-3}$$

Since both sides are $\log_5(\quad)$, the terms inside the logs must be equal

$$\therefore y = 2^{-3}$$

$$y = \frac{1}{8}$$

$$\therefore 5^{-3\log_5 2} = \frac{1}{8}$$

Turns out: $b^{\log_b x} = x$

we will see the proof next!

Ex. Show that $b^{\log_b x} = x$

Let the expression, $b^{\log_b x}$, equal to y

$$b^{\log_b x} = y$$

Re-write in log form

Base = b number = y exponent = $\log_b x$

$$\log_b y = \log_b x$$

Isolate y

$$\log_b y = \log_b x$$

Since both sides are $\log_{\mathrm{b}}(\quad)$, the terms inside the logs must be equal

$$\therefore y = x$$

Since $y = b^{\log_b x}$

$$\therefore b^{\log_b x} = x$$

Ex. Write $\log 75$ in terms of $\log 3$ and $\log 2$.

Break apart $\log 75$ into log of prime numbers

$$= \log 3 + \log 25$$

$$= \log 3 + \log 5^2$$

$$= \log 3 + 2\log 5$$

How to convert $\log 5$ into something with $\log 2$?

Need to be creative!

$\log 5 + \log 2 = \log 10$

$\therefore \log 5 = \log 10 - \log 2$

Sub back into original equation:

$$= \log 3 + 2(\log 10 - \log 2)$$

$$= \log 3 + 2 - 2\log 2$$

Challenging question:

Ex. Given $\log 8 = x$ and $\log 9 = y$, write $\log_5 24$ in terms of x and y.

$\log 8 = x$ $\qquad\qquad$ $\log 9 = y$

$\log 2^3 = x$ $\qquad\qquad$ $\log 3^2 = y$

$3\log 2 = x$ $\qquad\qquad$ $2\log 3 = y$

$\log 2 = \frac{x}{3}$ $\qquad\qquad$ $\log 3 = \frac{y}{2}$

$\log_5 24$

$= \frac{\log 24}{\log 5}$

$= \frac{\log 8 + \log 3}{\log 5}$

$= \frac{\log 2^3 + \log 3}{\log \frac{10}{2}}$ $\qquad$ Note: $\qquad \log 5 = \log \frac{10}{2}$

$= \frac{3\log 2 + \log 3}{\log 10 - \log 2}$

$= \frac{x + \frac{y}{2}}{1 - \frac{x}{3}}$

$= \frac{x + \frac{y}{2}}{1 - \frac{x}{3}} \cdot \frac{6}{6}$

$= \frac{6x + 3y}{6 - 2x}$

5.3 Homework:
3-5 bcf…

5.4 - Exponential and Logarithmic Equations

Key techniques to use:

1. Convert to the other form (exponential/logarithmic)

2. Log both sides of the equation (un-log both sides)

Ex. Solve for x.

a. $3^x = 81$

Re-writing with common base

$3^x = 3^4$

$x = 4$

Converting to logarithmic form

$x = \log_3 81$

$x = \log_3 3^4$

$x = 4\log_3 3$

$x = 4$

b. $3^x = 56$

Re-writing with common base

$3^x = 3^?$

Pretty much impossible…

Instead, solve using logarithms

$3^x = 56$

$x = \log_3 56$ or $x \approx 3.66$

(this is ok for now)

Better solution:

$x = \log_3 7 + \log_3 8$

$x = \log_3 7 + 3\log_3 2$

c. $\log(x+3) + \log x = 1$

$\log(x^2 + 3x) = 1$

$x^2 + 3x = 10^1$

$x^2 + 3x - 10 = 0$

$(x+5)(x-2) = 0$

$x = -5, 2$

Reject $x = -5$ because it will make the inside of the log negative

$\therefore x = 2$

A small change after the second step will lead to a different solution; same answer.

Alternatively,

$\log(x+3) + \log x = 1$

$\log(x^2 + 3x) = 1$

$\log(x^2 + 3x) = \log 10$

$x^2 + 3x = 10$

$\vdots$

$x = 2$

When both sides of the equation are both logarithms with the same base, it is now possible to "drop" the logs ("unlog" both sides).

Similar to the exponential rule, $b^m = b^n$ if and only if $m = n$

The rule for logarithms, $\log_{\mathrm{b}} m = \log_{\mathrm{b}} n$ if and only if $m = n$

Ex. Solve $8^{2x+3} = 12^{2x}$ Exact answer only.

When in doubt, log it out! (Log both sides of the equation)

$$\log 8^{2x+3} = \log 12^{2x}$$

$$(2x+3)\log 8 = 2x \log 12$$

$$2x\log 8 + 3\log 8 = 2x\log 12$$

$$2x\log 8 - 2x\log 12 = -3\log 8$$

$$x(2\log 8 - 2\log 12) = -3\log 8$$

$$x = \frac{-3\log 8}{2\log 8 - 2\log 12}$$

Leave answer where the inside of the logs are prime numbers.

$$x = -\frac{3\log 2^3}{2\log 2^3 - (2\log 3 + 2\log 4)}$$

$$x = -\frac{9\log 2}{6\log 2 - 2\log 3 - 2\log 2^2}$$

$$x = -\frac{9\log 2}{6\log 2 - 2\log 3 - 4\log 2}$$

$$x = -\frac{9\log 2}{2\log 2 - 2\log 3} \quad \text{or} \quad x = \frac{9\log 2}{2\log 3 - 2\log 2}$$

Ex. Solve $2(6)^{x+2} = 3^{2x-3}$. Exact answer only.

Log both sides

$$\log 2(6)^{x+2} = \log 3^{2x-3}$$

$$\log 2 + \log 6^{x+2} = \log 3^{2x-3}$$

$$\log 2 + (x+2)\log 6 = (2x-3)\log 3$$

$$\log 2 + x\log 6 + 2\log 6 = 2x\log 3 - 3\log 3$$

$$x\log 6 - 2x\log 3 = -3\log 3 - \log 2 - 2\log 6$$

$$x(\log 6 - 2\log 3) = -(3\log 3 + \log 2 + 2\log 6)$$

$$x = -\frac{3\log 3 + \log 2 + 2\log 6}{\log 6 - 2\log 3}$$

$$x = -\frac{3\log 3 + \log 2 + 2\log 2 + 2\log 3}{\log 2 + \log 3 - 2\log 3}$$

$x = -\frac{5\log 3 + 3\log 2}{\log 2 - \log 3}$ or $x = \frac{5\log 3 + 3\log 2}{\log 3 - \log 2}$

Ex. Solve $x^{\log x} = 100x$

Convert to log form

$$\log_x 100x = \log x$$

$$\frac{\log 100x}{\log x} = \log x$$

$$\log 100x = (\log x)^2$$

$$\log x + \log 100 = (\log x)^2$$

$$\log x + 2 = (\log x)^2$$

Let $y = \log x$

$$y + 2 = y^2$$
$$0 = y^2 - y - 2$$
$$0 = (y+1)(y-2)$$
$$y = -1, 2$$

$\log x = -1$ $\qquad$ $\log x = 2$

$x = \frac{1}{10}$ $\qquad$ $x = 100$

$$\therefore x = \frac{1}{10}, 100$$

Ex. Solve $x^{\log x} = 100x$

$$\log x^{\log x} = \log 100x$$

$$\log(x) \cdot \log(x) = \log 100 + \log x$$

$$(\log x)^2 = 2 + \log x$$

$$(\log x)^2 - \log x - 2 = 0$$

$$(\log x - 2)(\log x + 1) = 0$$

$\log x - 2 = 0$	$\log x + 1 = 0$
$\log x = 2$	$\log x = -1$
$x = 10^2$	$x = 10^{-1}$
$x = 100$	$x = \frac{1}{10}$

Ex. Solve for A in terms of B and C for $2\log A - \log B = C$.

$\log A^2 = \log B + C \log 10$	$\log A^2 - \log B = C$
$\log A^2 = \log B + \log 10^C$	$\log \frac{A^2}{B} = C$
$\log A^2 = \log B \cdot 10^C$	$\frac{A^2}{B} = 10^C$
$A^2 = B \cdot 10^C$	$A^2 = B \cdot 10^C$

$$A = \pm\sqrt{B \cdot 10^C}$$

Reject the negative because A cannot be negative

$$A = \sqrt{B \cdot 10^C}$$

Ex. If $\log 3 = a$ and $\log 8 = b$, determine $\log 18$ in terms of a and b.

$\log 3 = a$

$\log 8 = b$

$\log 2^3 = b$

$3 \log 2 = b$

$\log 2 = \frac{b}{3}$

$\log 18$

$= \log 9 + \log 2$

$= \log 3^2 + \log 2$

$= 2 \log 3 + \log 2$

$= 2a + \frac{b}{3}$

Or

$= \frac{6a+b}{3}$

5.4 Homework

1-7 bcf

5.5 Applications of Exponential and Logarithmic Functions

Ex. Estimate the time required for \$5000 to grow to \$30000 if it is invested at 10% compounded: (Round to 2 decimal places)

a. Monthly

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

$$30000 = 5000\left(1 + \frac{0.1}{12}\right)^{12t}$$

$$6 = \left(\frac{121}{120}\right)^{12t}$$

$$12t = \log_{\frac{121}{120}} 6$$

$$t = \frac{\log_{\frac{121}{120}} 6}{12} \text{ or } \frac{1}{12}\log_{\frac{121}{120}} 6 \qquad (\log 6 \div \log(121 \div 120)) \div 12$$

$t = 17.99$ years

b. Continuously

$A = Pe^{rt}$ For investment problems: $A = A_0e^{kt} \rightarrow A = Pe^{rt}$

$$30000 = 5000e^{0.1t}$$

$$6 = e^{0.1t}$$

$\log_e 6 = 0.1t$ Recall: $\log_e x = \ln x$

$$\ln 6 = 0.1t$$

$$t = \frac{\ln 6}{0.1}$$

$t = 17.92$ years

Ex. The half-life of daniezhanium-613 is 13 years. Find the time for 80% of a 5 gram sample to decay. Round to 2 decimal places.

80% to decay → 20% remains

Find amount remaining.

∴ 5 grams → 1 gram

Use growth/decay formula to solve for t.

$$A = A_0 x^{\frac{t}{T}}$$

$$1 = 5\left(\frac{1}{2}\right)^{\frac{t}{13}}$$

$$\frac{1}{5} = \left(\frac{1}{2}\right)^{\frac{t}{13}}$$

$$\frac{t}{13} = \log_{0.5}\left(\frac{1}{5}\right)$$

$$t = 13\log_{0.5}\left(\frac{1}{5}\right)$$

$$t = 30.19 \text{ years}$$

∴ The 5 gram sample of daniezhanium-613 needs to take 30.19 years to decay to 1 gram.

Ex. The population of undead penguins is currently at 12000. It is estimated that 28 weeks later, the population of these undead penguins will reach 144000. Determine the population 40 weeks from now. Round to the nearest day.

Use continuous growth formula: $A = A_o e^{kt}$

First solve for growth constant k, with provided information provided.

$$144000 = 12000e^{k(28)}$$

$$12 = e^{28k}$$

$$28k = \ln 12$$

$$k = \frac{1}{28}\ln 12$$

Now we can write the growth equation as:

$$A = 12000e^{\left(\frac{1}{28}\ln 12\right)t}$$

Now to find the population after 40 weeks.

$$A = 12000e^{\left(\frac{1}{28}\ln 12\right)(40)}$$

$$A = 417704.1$$

$$A \approx 417704$$

The population of undead penguins will reach 417704 after 40 weeks.

5.5 Homework

2-4, 6, 7, 10, 11