Chapter 9 - Combinatorics

9.1 Combinatorics

Combinatorics is the branch of mathematic that is primarily concerned with counting.

Fundamental Counting Principle (The Multiplication Principle)

- If we can perform a first task in a different ways
- If we can perform a second task in b different ways
- If we can perform a third task in c different ways, and so on…

Then the first task followed by the second and so on can be performed in

$\boldsymbol{a \cdot b \cdot c}$… different ways

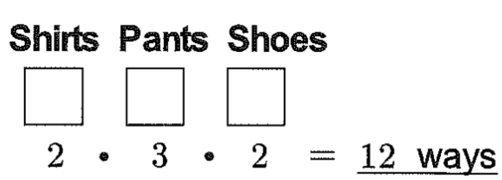
These tasks do not affect each other; they are called **independent events**.

Examples Using the Multiplication Principle

Ex. A man has 2 shirts (red and yellow), 3 pairs of pants (black, white, and navy), and 2 pairs of shoes (green and orange). How many different outfits can he wear?

There are 2 choices of shirts. For each choice of shirts, there are 3 choices of pants. So, there are 2×3 possible combinations for shirts and pants. For each combination of shirts and pants, there are 2 choices of shoes.

Therefore, the possible number of outfits is $2 \times 3 \times 2 = 12$



Ex. A particular automobile has 4 different models, 3 sizes of motors and 6 colour schemes. How many different ways can an automobile be ordered?

Models Motors Colours

$$4 \times 3 \times 6 = 72$$

There are 72 different ways

Ex. The first 4 questions on a quiz are true-false questions, while the next 6 questions are multiple choice with possible answers a, b, c, d, and e. How many different possible answer sequences are there for these 10 questions?

T/F: 2 x 2 x 2 x 2 = 2^4 = 16

M/C: 5 x 5 x 5 x 5 x 5 x 5 = 5^6 = 15625

Total = T/F $\times$ M/C

= 16 $\times$ 15625

= 250000

250,000 possible sequences

Ex. How many telephone numbers are available with the 778 prefix?

After the 778 prefix, each of the 7 numbers have 10 possibilities, assuming no restrictions.

7 7 8 - ___ ___ ___ ___ ___ ___ ___

#1 #2 #3 #4 #5 #6 #7

$$10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10$$

$10^7 = 10{,}000{,}000$ possible phone numbers

Ex. How many different ways can 5 different books be arranged on a shelf?

The first book would have 5 choices. With 4 books left, the second choice would have 4 choices. Then the third book with have 3 choices, while the fourth and fifth books would have 2 and 1 choices respectively.

$5 \times 4 \times 3 \times 2 \times 1 = 120$ different ways

Ex. How many different can 3 books be arranged on a shelf, using 5 different books?

$5 \times 4 \times 3 = 60$ different ways

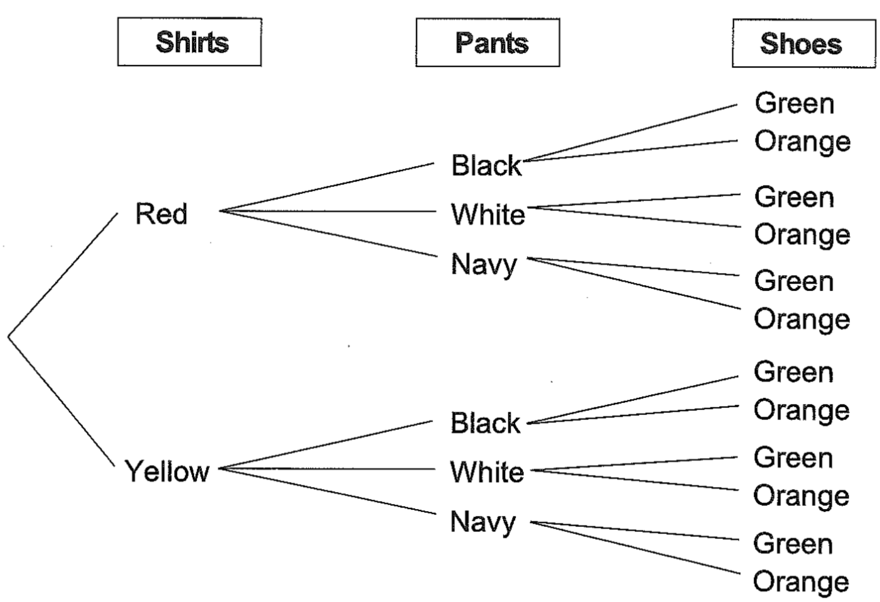
Tree Diagrams

We use tree diagrams to have a systematic way of counting the outcomes.

Examples Using Tree Diagrams

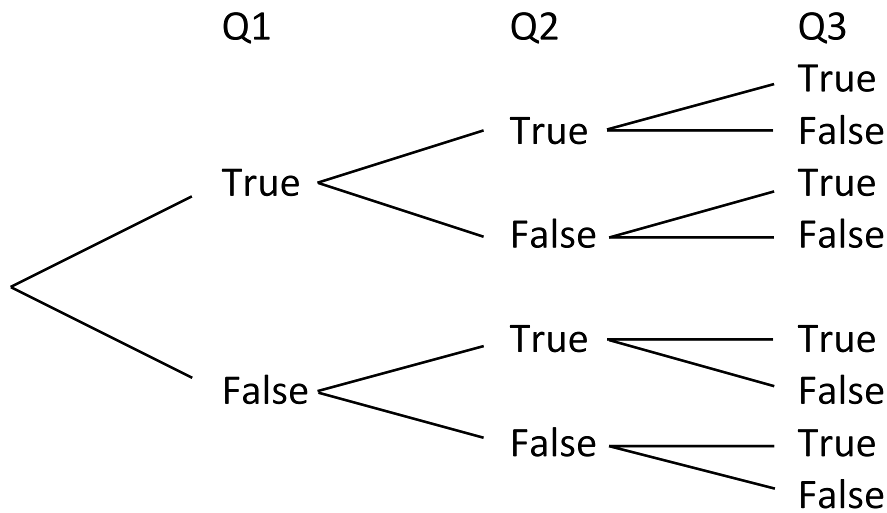
Ex. A man has 2 shirts (red and yellow), 3 pairs of pants (black, white, and navy), and 2 pairs of shoes (green and orange). How many different outfits can he wear?

List out possible outcomes using a tree diagram



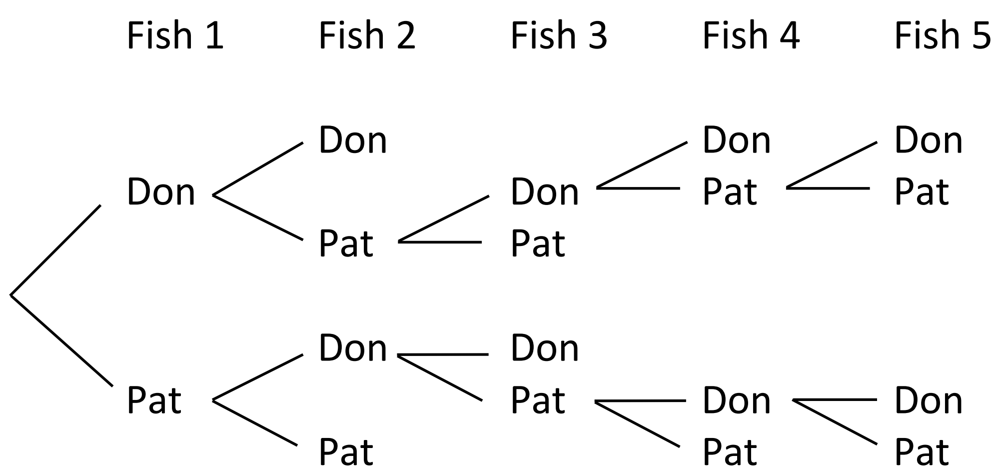
$\therefore$ 12 ways

Ex. How many ways can a 3 item true-false test be answered?



∴ 8 ways

Ex. Don and Pat are in a fishing tournament. The first person to catch 2 fish in a row or 3 fish in total wins the tournament. How many different outcomes are possible?



∴ 10 ways

Outcomes:

DD, PP, DPP, PDD, DPDD, PDPP, DPDPD, DPDPP, PDPDD, PDPDP

Factorial Notation!

The product of the consecutive positive integers from 1 to n is given a special name, n factorial, which is written $n!$ $(n \geq 0)$

$0! = 1$ by definition

$1! = 1$

$2! = 2 \times 1 = 2$

$3! = 3 \times 2 \times 1 = 6$

$4! = 4 \times 3 \times 2 \times 1 = 24$

$\vdots$

$$n! = n(n-1)(n-2)\dots 3 \cdot 2 \cdot 1$$

Ex. Evaluate 5!

$5!$

$= 5 \times 4 \times 3 \times 2 \times 1$

$= 120$

Ex. How many different "words" can be made using all the letters in abcde?

This is the exact same situation as the problem with arranging 5 books. First book has 5 choices, second book as 4 choices, and etc.

$5! = 120$

There are 120 "words" that can be constructed from abcde

Ex. A three-digit password is made from using numbers 0-9.

a. How many possible passwords are there if there are no restrictions

$$10 \cdot 10 \cdot 10 \cdot 10$$

$$= 10000$$

b. How many possible passwords are there if none of the numbers be repeated?

$$10 \cdot 9 \cdot 8 \cdot 7$$

$$= 5040$$

c. How many possible passwords are there if a number can be used up to 3 times?

$$10 \cdot 10 \cdot 10 \cdot 10 - 10$$

$$= 9990$$

Ex. Simplify and the evaluate $\frac{86!}{84!}$

$\frac{86!}{84!}$ Re-write $86!$ as $86 \times 85 \times 84!$

$= \frac{86 \times 85 \times 84!}{84!}$ Cancel out $84!$ from the numerator and denominator

$= 86 \times 85$

$= 7310$

Ex. Simplify and then evaluate $\frac{20!-18!}{18!}$

$\frac{20!-18!}{18!}$ Factor out 18!

$= \frac{18!(20\times19-1)}{18!}$ Reduce 18! from numerator and denominator

$= 20 \times 19 - 1$

$= 380 - 1$

$= 379$

Ex. Simplify $\frac{(n+1)!}{(n-1)!}$

$\frac{(n+1)!}{(n-1)!}$

$= \frac{(n+1)(n)(n-1)!}{(n-1)!}$

Need to re-write the larger factorial so it has a common factor with the smaller factorial $(n+1)! = (n+1)(n)(n-1)!$

$= (n+1)(n)$

$= n^2 + n$

Ex. Simplify $\frac{n!}{(n-2)!+(n-3)!}$

$= \frac{n(n-1)(n-2)(n-3)!}{(n-2)(n-3)!+(n-3)!}$

$= \frac{n(n-1)(n-2)}{(n-2)+1} = \frac{n(n-1)(n-2)}{n-1}$

$= n(n-2) = n^2 - 2n$

Solving Factorial Equations

Ex. Solve $\frac{n!}{(n-2)!} = 3!$

$\frac{n(n-1)(n-2)!}{(n-2)!} = 6$

$n(n-1) = 6$

$n^2 - n - 6 = 0$

$(n+2)(n-3) = 0$

$n = -2, 3$ reject -2

$n = 3$

Ex. Solve $3!\,(n+1)! = 5!\,(n-1)!$

$6(n+1)(n)(n-1)! = 120(n-1)!$

$(n+1)(n) = 20$ able to cancel out $(n-1)!$, because it $= 1$

$n^2 + n - 20 = 0$

$(n+5)(n-4) = 0$

$n = -5, 4$ reject -5

$n = 4$

9.1 Homework

7.1 pg 451 # 2, 3, 5, 8, 9, 12, 16, 17bd, 18bcf, 19

9.2 Permutations, $n \text{ P } r$

A permutation is the fundamental counting principle put into a formula.

Permutations (Rule A)

Permutations (Rule A) must follow the following three restrictions:

1. The n objects are all different

2. No object can be repeated

3. Order makes a difference (ex. xy is different from yx)

A permutation is the arrangement of r objects with ($r \leq n$) can be written as $P(n, r) = n\text{P}r$

$$\text{n P r} = \frac{n!}{(n-r)!}$$

Note: Combinations and Permutations are similar but have a distinctive difference.

Permutations – order matters

Combinations – order does not matter

For example, **permutation** considers 538-2783 as being **different** from 583-2783, while **combinations** recognize the two as the **same**.

Another example, consider three people, Chris, Daryl, and Emma.

In a three-person committee, **combinations** considers these three people as one combination.

For a three-person executive committee with a president, vice-president, and treasurer, **permutations** would consider these people as ($3 \text{ P } 3 = 3!$) 6 permutations.

Ex. How many different 3-digit numbers can be made using the numbers 1 through 7 by only using each number once?

$$7P3$$

$$= \frac{7!}{(7-3)!}$$

$$= \frac{7\times6\times5\times4!}{4!}$$

$$= 7 \times 6 \times 5 \; = \; 210$$

There are 210 permutations.

Ex. How many ways can a president, vice president and treasurer be selected from a class of 25 students?

$$25P3$$

$$= \frac{25!}{22!}$$

$$= 25 \times 24 \times 23$$

$= 13800$ ways

Ex. 8 friends are going to watch a movie and want to sit together. In a row with 8 seats, how many ways can they pick their seats?

$$8\,P\,8$$

$$= 8!$$

$$= 40320$$

Ex. 8 friends are going to dinner and sit an 8-person round table. How many ways can they sit together?

Because the table is round, the first person only has 1 choice (to sit in an open seat). The second person has 7 choices to sit relative to the first person. The third person has 6 choices relative to the first person… $7\,P\,7$

$= 7!$

$= 5040$

Ex. 8 friends are going to watch a movie and want to sit together. One couple always have to sit together. In a row with 8 seats, how many ways can they pick their seats?

Treat the couple as one person, so it's like 7 people. But the couple have 2 sitting arrangements.

$2\text{ P }2 \cdot 7\text{ P }7$

$= 2! \cdot 7!$

$= 10080$

Ex. 8 friends are going to watch a movie and want to sit together. The couple from the previous question broke up and cannot sit together. In a row with 8 seats, how many ways can they pick their seats?

The total ways of 8 people sitting is $8! = 40320$

The total ways the couple sit together is $2! \cdot 7! = 10080$

The total ways the broke up couple avoid each other is the difference between the previous two sums $= 40320 - 10080 = 30240$

Permutations (Rule B)

n objects where there are some objects that are the same

Comparing Permutations Rule A and B

Rule A: $abcd$ Rule B: $aabc$

Permutations (Rule B)

In the example above, $aabc$, we could consider the same objects to be different, so a_1a_2bc. So, the arrangement a_1a_2bc looks different from a_2a_1bc, but the two a's indistinguishable. Thus, the two arrangements are considered the same.

In order to not count the repeats, we divide the total possible arrangements by the factorial of each repeat.

So, arrangements for $aabc$ would be $\frac{4!}{2!} = 12$

Ex. How many different 5-letter "words" can be formed from the letters "SWEET"?

E occurs twice, need to divide the possible outcomes by 2!

$\frac{5!}{2!} = 60$

Ex. How many different words can be formed from BANANA using all the letters?

There are multiple letters that repeat. A occurs 3 times, while N occurs twice. Need to divide the total by 3! and 2!.

$\frac{6!}{3!2!} = 60$

Ex. How many different words can be formed from MISSISSIPPI?

11 total letters, I occurs 4 times, S occurs 4 times, while P occurs twice.

$$\frac{11!}{4!4!2!} = 34650$$

Ex. Find the number of different ways of placing 16 balls in a row given that 4 are black, 3 are white, 7 are red and 2 are blue.

$$\frac{16!}{4!3!7!2!} = 14414400$$

9.2 Homework

1-21 odd

9.3 Combinations, ${}_nC_r$

Combinations are similar to permutations, where they both look for possible ways/outcomes to accomplish a task.

Combinations are arrangements of r number of objects chosen from n number of objects where:

- n objects are all different
- no objects can be repeated
- order does **NOT** make a difference
  (ab and ba are the same combination)

The number of combinations choosing r objects from n objects is given by:

$${}_nC_r = \frac{n!}{r!(n-r)!} \qquad \text{where } r \leq n$$

Other ways to write combinations:

$$n\,C\,r \quad = \quad C(n,r) \quad = \quad \binom{n}{r}$$

Combination Examples

Ex. How many ways can 2 people be selected from a group of 6 people?

Order does not matter $\quad n = 6 \quad r = 2$

$${}_6C_2$$

$$= \frac{6!}{2!(6-2)!}$$

$$= \frac{6!}{2!4!}$$

$$= \frac{720}{2 \times 24} = 15$$

There are 15 ways to select 2 people from a group of 6.

Ex. How many 5 card hands are possible in a regular deck of 52 cards?

Order does not matter $n = 52$ $r = 5$

${}_{52}C_5$

$= \frac{52!}{5!47!}$

$= \frac{52\times51\times50\times49\times48\times47!}{5!47!}$

$= 2{,}598{,}960$

There are 2,598,960 different possible hands.

Ex. How many different tickets are possible when playing Lotto 6/49? (must pick 6 numbers out of 49 numbers in any order)

${}_{49}C_6$

$= \frac{49!}{6!43!}$

$= \frac{49\times48\times47\times46\times45\times44\times43!}{6!43!}$

= 13,983,816

There are 13,983,816 possible combinations

Ex. From 6 students and 4 teachers, a committee of 2 students and 2 teachers must be chosen. How many ways can this be done?

(number of ways to choose students)(number of ways to choose teachers)

${}_6C_2 \times {}_4C_2$

= 15 x 6 = 90

There are 90 possible ways

Ex. If night school offers 100 courses, 8 of which are in Mathematics, and you select 4 courses by random selection, how many possibilities include one Mathematics course?

$_8C_1 \times {}_{92}C_3$

$$\binom{8}{1}\binom{92}{3} = 1{,}004{,}640$$

There are $1{,}004{,}640$ possibilities.

Solving Combination Equations

Ex. Solve for n. $_nC_2 = 66$

$$\frac{n!}{2!(n-2)!} = 66$$

$$\frac{n(n-1)(n-2)!}{2(n-2)!} = 66$$

$$n^2 - n = 132$$

$$n^2 - n - 132 = 0$$

$$(n-12)(n+11) = 0$$

$$n = 12, -11$$

$$n = 12$$

9.3 Homework

2, 5-7, 9, 11

9.4 Binomial Theorem

In previous grade levels of mathematics, you learned how to find the following binomial expansions:

$(x+y)^1$

$= x+y$

$(x+y)^2$

$= x^2+2xy+y^2$

$(x+y)^3$

$= (x^2+2xy+y^2)(x+y)$

$= x^3+3x^2y+3xy^2+y^3$

What would $(x+y)^4$ equal to?

$(x+y)^4$

$= (x+y)^2(x+y)^2$

$= (x^2+2xy+y^2)(x^2+2xy+y^2)$

$= x^4+4x^3y+6x^2y^2+4xy^3+y^4$

This is tedious and can become very complex at a higher degree.

Is there a more efficient way to compute binomial expansions?

Yes, using binomial theorem!

Binomial Theorem and Pascal's Triangle

The coefficients of a binomial expansion are the same as the numbers in Pascal's Triangle.

Pascal's Triangle	Sum of row
1	$2^0 = 1$
1 1	$2^1 = 2$
1 2 1	$2^2 = 4$
1 3 3 1	$2^3 = 8$
1 4 6 4 1	$2^4 = 16$
1 5 10 10 5 1	$2^5 = 32$
1 6 15 20 15 6 1	$2^6 = 64$

6 is the sum of 1 and 5 the numbers above.

Binomial Expansion

$$(x+y)^0 = 1$$

$$(x+y)^1 = x+y$$

$$(x+y)^2 = x^2+2xy+y^2$$

$$(x+y)^3 = x^3+3x^2y+3xy^2+y^3$$

$$(x+y)^4 = x^4+4x^3y+6x^2y^2+4xy^3+y^4$$

$$(x+y)^5 = x^5+5x^4y+10x^3y^2+10x^2y^3+5xy^4+y^5$$

$$(x+y)^6 = x^6+6x^5y+15x^4y^2+20x^3y^3+15x^2y^4+6xy^5+y^6$$

Notice the numbers in Pascal's Triangle and coefficients in the binomial expansion.

In general, binomial expansions have the form:

$$(x+y)^n = {}_nC_0 x^n y^0 + {}_nC_1 x^{n-1} y^1 + {}_nC_2 x^{n-2} y^2 + \cdots + {}_nC_n x^{n-n} y^n$$

Compute Binomial Expansion Using Binomial Theorem

Ex. Expand $(x+y)^4$

$$= {}_4C_0 x^4 y^0 + {}_4C_1 x^3 y^1 + {}_4C_2 x^2 y^2 + {}_4C_3 x^1 y^3 + {}_4C_4 x^0 y^4$$

$$= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

Ex. Expand $(x+2)^5$

$$= {}_5C_0 x^5 2^0 + {}_5C_1 x^4 2^1 + {}_5C_2 x^3 2^2 + {}_5C_3 x^2 2^3 + {}_5C_4 x^1 2^4 + {}_5C_5 x^0 2^5$$

$$= x^5 + 10x^4 + 40x^3 + 80x^2 + 80x + 32$$

Finding a Specific Term of a Binomial Expansion

Instead of finding all the terms, it is possible to find a specific term in a binomial expansion.

For $(a+b)^n$, the specific term at position k (where t_1 has $k=0$) is given by:

$$t_{k+1} = {}_nC_k (a)^{n-k} \cdot b^k$$

Note: First term t_1 has combination ${}_nC_0$ and not ${}_nC_1$

Ex. Find the 5th term of $(2x + y)^{12}$

First, find k for t_5 and n.

$t_5 = t_{k+1} \quad \rightarrow \quad k = 4$ and $n = 12$

$$t_5 = {}_{12}C_4(2x)^{12-4}y^4$$

$$t_5 = 495(256x^8)(y^4)$$

$$t_5 = 126720x^8y^4$$

Ex. Find the coefficient of x^4 in the expansion of $\left(\sqrt{x} - 2\right)^{10}$.

$t_{k+1} = \binom{10}{k}\left(\sqrt{x}\right)^{10-k}(-2)^k = \mathrm{C}x^4$ for some constant C

Solve for k by comparing $\binom{10}{k}\left(\sqrt{x}\right)^{10-k}(-2)^k = \mathrm{C}x^4$

$$\left(x^{\frac{1}{2}}\right)^{10-k} = x^4$$

$$(x)^{5-\frac{1}{2}k} = x^4$$

$$5 - \frac{1}{2}k = 4$$

$$-\frac{1}{2}k = -1$$

$$k = 2$$

To find the coefficient

$$t_{2+1} = \binom{10}{2}\left(\sqrt{x}\right)^{10-2}(-2)^2 = 180x^4$$

∴ The coefficient is 180

9.4 Homework:
1-13 odd