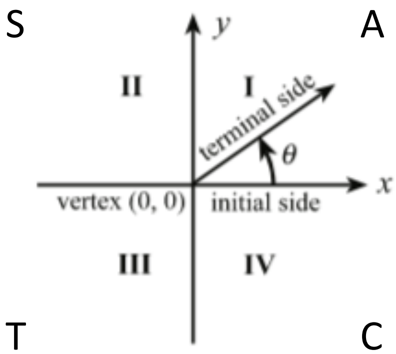
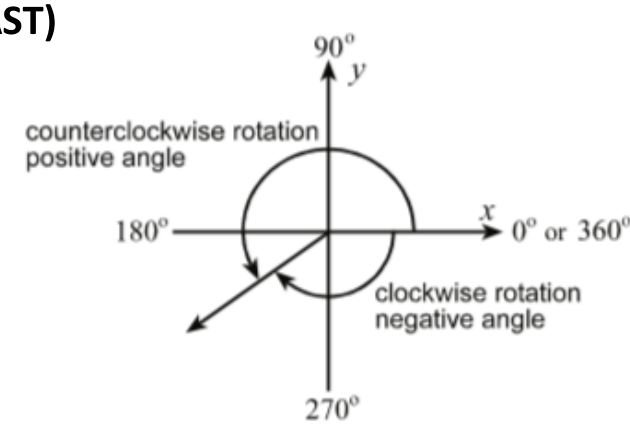
Ch 6 – Trigonometry PI (Functions + Equations)

6.1 Trigonometric Functions

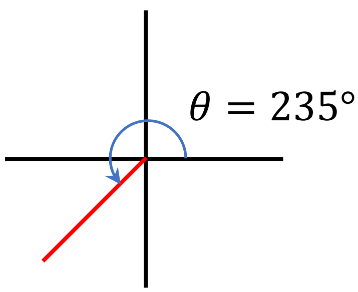
Angles in Standard Position (CAST)

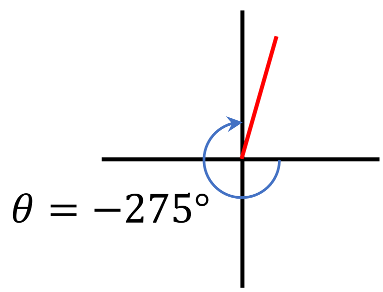




Examples of Degree Measures

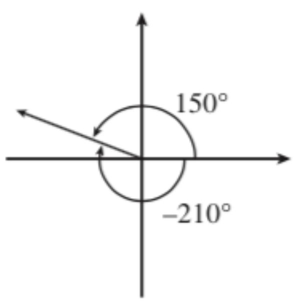
$\theta = 235°$ $\theta = -275°$



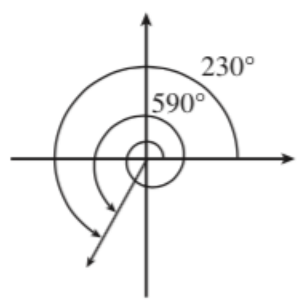


Co-terminal Angles

- Angles that share the same terminal arm



150° and –210° are coterminal angles



230° and 590° are coterminal angles

To find co-terminal angles, you need to add or subtract 360° from the initial angle.

Ex. Determine a positive and a negative coterminal angle for 111°.

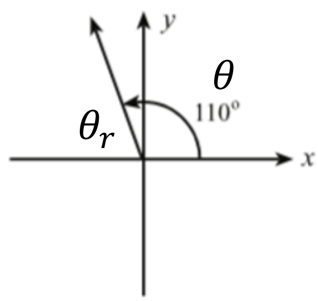
Positive: $111° + 360° = 471°$

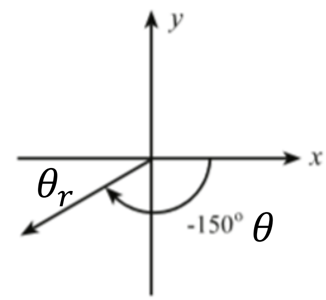
Negative: $111° - 360° = -249°$

Reference Angles, $\boldsymbol{\theta_r}$

A reference angle is the angle from the terminal arm to the nearest horizontal side.

Reference angles are always between 0° and 90°





When θ is $110°$,
the reference angle, θ_r, is $70°$

When θ is $-150°$,
the reference angle, θ_r, is $30°$

Ex. Determine the reference angle given θ.

a. $\theta = 62°$

$\theta_r = 62°$

In QI, $\theta_r = \theta$

b. $\theta = 120°$

$\theta_r = 180 - 120°$

In QII, $\theta_r = 180° - \theta$

$\theta_r = 60°$

c. $\theta = 191°$

$\theta_r = 191° - 180°$

In QIII, $\theta_r = \theta - 180°$

$\theta_r = 11°$

d. $\theta = 312°$

$\theta_r = 360 - 312°$

In QIV, $\theta_r = 360° - \theta$

$\theta_r = 48°$

Radians and Degrees

Radians are a more natural way of describing angles; they are the standard unit of angular measure in many areas of mathematics. A radian is an arc length of a circle and are dimensionless.

A circle with a radius of 1 unit has a circumference of 2π units; going through a full circle is 2π radians. In degrees, the same circle would have a full rotation of 360°.

From the above statement we can conclude that: $2\pi = 360°$

With $2\pi = 360°$, dividing both sides would yield: $\pi = 180°$

Common radian and degree equivalents

$2\pi = 360°$	$\pi = 180°$	$\frac{\pi}{2} = 90°$
$\frac{\pi}{3} = 60°$	$\frac{\pi}{4} = 45°$	$\frac{\pi}{6} = 30°$

Converting from Degrees to Radians

Use $\frac{\pi}{180°}$ as the conversion factor.

Ex. Determine the following angles in radians (exact value).

a. $60°$

$= 60° \times \frac{\pi}{180°}$

$= \frac{60}{180}\pi$

$= \frac{1}{3}\pi$ or $\frac{\pi}{3}$ radians

b. $155°$

$= 155° \times \frac{\pi}{180°}$

$= \frac{155\pi}{180}$

$= \frac{31\pi}{36}$ radians

Ex. Determine the following angles in radians (to 3 decimal places)

a. $156°$

$= 156° \times \frac{\pi}{180°}$

$= 2.723$ radians

b. $260°$

$= 260° \times \frac{\pi}{180°}$

$= 4.538$ radians

Converting from Radians to Degrees

Use $\frac{180°}{\pi}$ as the conversion factor.

Ex. Determine following angle in degrees (exact value).

a. $\frac{3\pi}{2}$

$= \frac{3\pi}{2} \times \frac{180°}{\pi}$

$= 270°$

b. $-\frac{7}{3}\pi$

$= -\frac{7\pi}{3} \times \frac{180°}{\pi}$

$= -420°$

Ex. Determine following angle in degrees (1 decimal place).

a. 2.72 radians

$= 2.72 \times \frac{180°}{\pi}$

$= 155.8°$

b. 5.19

$= 5.19 \times \frac{180°}{\pi}$

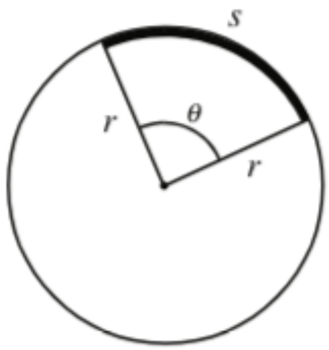
$= 297.4°$

Arc Length, $S = r\theta$

The length around a segment of a circle.

Ex. Show that the arc length s, is equal to $r\theta$.

A central angle is the angle that forms when two radii meet at the center of a circle.



θ = central angle in radians
r = radius
s = arc length

A full circle, has a central angle of 2π and arc length of $2\pi r$ (circumference)
A half circle, has a central angle of π and arc length of πr
A quarter circle, has a central angle of $\frac{\pi}{2}$ and arc length of $\frac{\pi r}{2}$

We see that arc length is proportional to central angle. Let's compare a full circle with radius r, to arc length s and central angle θ.

$$\frac{central\ angle}{arc\ length} = \frac{1\ full\ turn\ in\ a\ circle}{circumference}$$

$$\frac{\theta}{s} = \frac{2\pi}{2\pi r}$$

$$\frac{\theta}{s} = \frac{1}{r}$$

$$s = r\theta$$

Note: Calculating arc length with central angle in degrees:

If $\theta = 60°$ and $r = 10$ cm, find the arc length

$S = 10$ cm $\cdot\ 60° = 600°$ cm??

In the formula $s = r\theta$, **$\boldsymbol{\theta}$ must be in radians**.

Ex. If $\theta = 60°$ and radius is 10 cm, determine the arc length.

First, convert the angle into radians

$$\theta = 60° \times \frac{\pi}{180°} = \frac{\pi}{3}$$

Next, determine the arc length

$$s = r\theta$$

$$s = 10 \times \frac{\pi}{3}$$

$$s = \frac{10\pi}{3} \text{ cm}$$

Ex. An arc has length of 20 cm and a radius of 8 cm, determine the central angle in degrees to 1 decimal place.

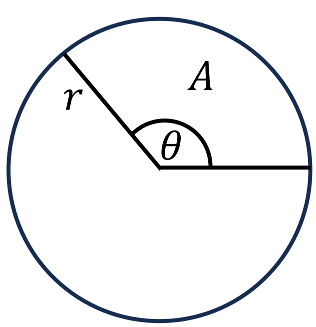
$$20 = 8\theta$$

$$\theta = \frac{20}{8}$$

$$\theta = \frac{5}{2} \cdot \frac{180°}{\pi} = 143.239° \approx 143.2°$$

Sector Area

The area of a sector is a part of the area of the full circle.



$$\frac{central\ angle}{sector\ area} = \frac{1\ full\ turn\ in\ a\ circle}{area\ of\ a\ circle}$$

$$\frac{\theta}{A} = \frac{2\pi}{\pi r^2}$$

$$\frac{\theta}{A} = \frac{2}{r^2}$$

$$A = \frac{r^2\theta}{2}$$

$A = \frac{r^2\theta}{2}$ where θ must be in radians.

Ex. Determine the sector area of a circle that has a radius of 8 in and central angle of 225°.

Convert angle to radians

$$\theta = 225° \times \frac{\pi}{180°} = \frac{5\pi}{4}$$

Find sector area

$$A = \frac{r^2\theta}{2}$$

$$A = \frac{8^2 \cdot \frac{5\pi}{4}}{2}$$

$$A = 40\pi \text{ in}^2$$

Ex. If a sector area is 70 cm^2 and has a central angle of $\frac{3\pi}{5}$, determine the radius to 1 decimal place.

$$A = \frac{r^2\theta}{2}$$

$$70 = \frac{r^2\left(\frac{3\pi}{5}\right)}{2}$$

$$\frac{700}{3\pi} = r^2$$

$$r = \sqrt{\frac{700}{3\pi}}$$ radius cannot be negative, so no need for $\pm$ sign.

$$r = 8.6 \text{ cm}$$

6.1 Homework

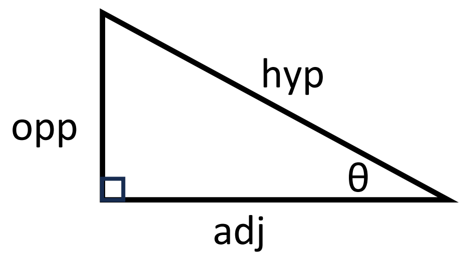
1-8 bcf…, 10, 12, 14, 16

6.2 Trigonometric Function of Acute Angles

Common Trigonometric Ratios, and Reciprocal Ratios

From previous math courses, we have seen the sine, cosine, and tangent ratios. The reciprocal of those three ratios occur often, and they are **cosecant**, **secant** and **cotangent**.

The trigonometric ratios for a right triangle:



$\sin\theta = \frac{opp}{hyp}$ $\quad$ $\csc\theta = \frac{1}{\sin\theta}$ $\rightarrow$ $\csc\theta = \frac{hyp}{opp}$ (cosecant)

$\cos\theta = \frac{adj}{hyp}$ $\quad$ $\sec\theta = \frac{1}{\cos\theta}$ $\rightarrow$ $\sec\theta = \frac{hyp}{adj}$ (secant)

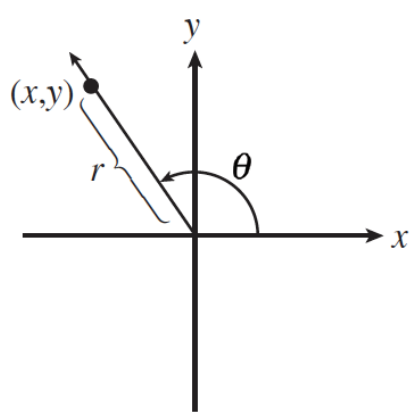
$\tan\theta = \frac{opp}{adj}$ $\quad$ $\cot\theta = \frac{1}{\tan\theta}$ $\rightarrow$ $\cot\theta = \frac{adj}{opp}$ (cotangent)

Coordinates and Trigonometric Ratios

Trigonometric ratios can be defined by a general coordinate (x, y) and the origin. Connect the two coordinates to form a terminal arm with length r.

$$r^2 = x^2 + y^2$$
$$r = \sqrt{x^2 + y^2}$$

Use Pythagorean Theorem to find r.
r is the length of the terminal arm.



$\sin\theta = \frac{y}{r}$ $\quad$ $\csc\theta = \frac{r}{y}$

$\cos\theta = \frac{x}{r}$ $\quad$ $\sec\theta = \frac{r}{x}$

$\tan\theta = \frac{y}{x}$ $\quad$ $\cot\theta = \frac{x}{y}$

Trigonometric Ratios, Quadrants, and Coordinates

Ex. What quadrant has $\sin\theta < 0$, and $\tan\theta < 0$?

Sine is negative in QIII and QIV, while tangent is negative in QII and QIV. They are both negative in QIV.

$\therefore$ Quadrant IV

Ex. Determine $\tan\theta$, if $\sec\theta = \frac{5}{2}$.

$\sec\theta$ is the reciprocal of $\cos\theta$, since $\sec\theta = \frac{5}{2}$, then $\cos\theta = \frac{2}{5}$

Since $\sec\theta > 0$, which means $\cos\theta > 0$. $\cos\theta$ is positive in QI and QIV, so there will be two solutions.

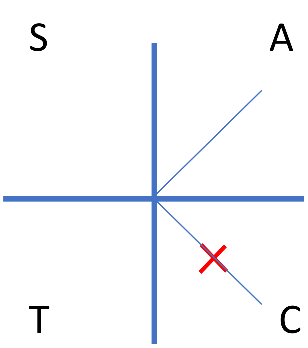
$\cos\theta = \frac{2}{5}$ tells us that: $x = 2$ and $r = 5$.

$\tan\theta$ requires x and y values, need to find y.

$$r^2 = x^2 + y^2$$
$$25 = 4 + y^2$$
$$y = \pm\sqrt{21}$$

$\tan\theta = \frac{\sqrt{21}}{2}, -\frac{\sqrt{21}}{2}$ These are ratios are in QI and QIV respectively.

Ex. Determine $\csc\theta$, if $\sec\theta = 2$ and $\cot\theta > 0$.



If $\sec\theta = 2$ then $\cos\theta = \frac{1}{2}$

$\therefore x = 1 \quad r = 2$

$y = \pm\sqrt{4-1} = \pm\sqrt{3}$

Since $\cot\theta > 0$, terminal arm could be in QI and QIII

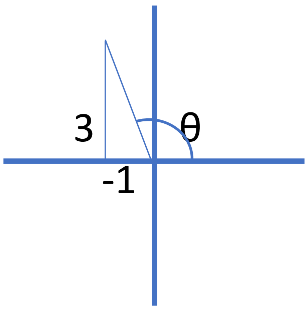
Since $\sec\theta$ is positive, the terminal arm cannot be in QIII; must be in QI.

$\therefore$ y cannot be negative

$\therefore \csc\theta = \frac{2}{\sqrt{3}}$ and reject $\csc\theta = -\frac{2}{\sqrt{3}}$

Trigonometric Ratios Defined by a Coordinate

Ex. Determine the 3 basic trig ratios for the angle constructed using the coordinate $(-1, 3)$ being on the terminal arm.



Since $x = -1$ and $y = 3$, find r.

$$r = \sqrt{x^2 + y^2}$$
$$= \sqrt{1 + 9} = \sqrt{10}$$

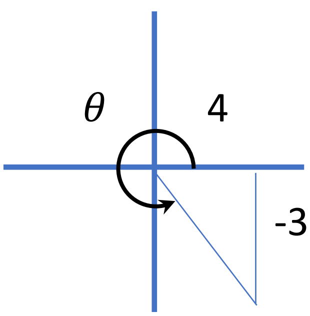
The three basic trig ratios are:

$$\sin\theta = \frac{3}{\sqrt{10}} \qquad \cos\theta = -\frac{1}{\sqrt{10}}$$

$$\tan\theta = -3$$

Trigonometric Ratios Defined by a Linear Equation

Ex. The terminal arm of an angle lies on the line $3x + 4y = 0$, and $x \geq 0$. Determine $\sin\theta$ and $\cos\theta$.



Graph $3x + 4y = 0$, convert to $y = mx + b$

$$4y = -3x$$
$$y = -\frac{3}{4}x \quad \text{with } x \geq 0$$

Since $x = 4$ and $y = -3$, find r.

$$r = \sqrt{16 + 9} = 5$$

$\sin\theta$ and $\cos\theta$ are:

$$\sin\theta = -\frac{3}{5} \qquad \cos\theta = \frac{4}{5}$$

Trig and Inverse Trig Expressions

Ex. Simplify the following expression.

a. $\sin\left(\cos^{-1}\frac{x}{6}\right)$

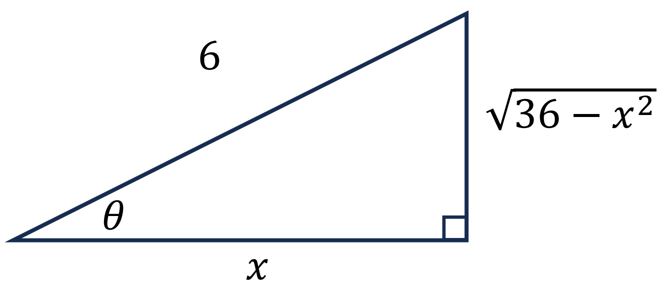
First, write a let statement to represent the angle $\cos^{-1}\frac{x}{6}$

Let $\theta = \cos^{-1}\frac{x}{6}$

Next, re-write the above statement as a cosine ratio

$\cos\theta = \frac{x}{6}$

Now, draw a right triangle where the adjacent side is x and hypotenuse is 6. Find an expression for the missing side using Pythagorean Theorem.



In the expression $\sin\left(\cos^{-1}\frac{x}{6}\right)$, replace $\cos^{-1}\frac{x}{6}$ with θ

$= \sin(\theta)$

Using the triangle above, find the sine ratio for θ

$= \frac{\sqrt{36-x^2}}{6}$

b. $\cot\left(\sec^{-1}\frac{a}{b}\right)$

Let $\theta = \sec^{-1}\frac{a}{b}$

$\sec\theta = \frac{a}{b} \qquad \rightarrow \qquad \cos\theta = \frac{b}{a}$

$x = b \qquad r = a \qquad \rightarrow y = \sqrt{a^2 - b^2}$

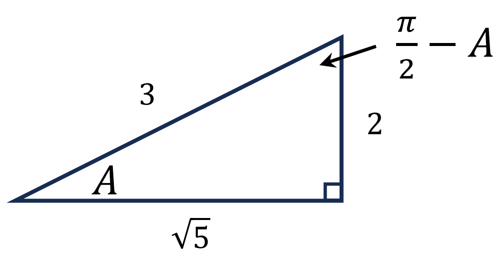
$= \cot(\theta)$

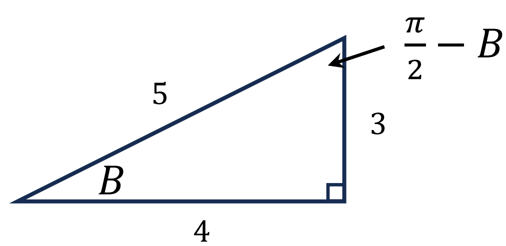
$= \frac{b}{\sqrt{a^2-b^2}}$

Ex. If $\csc A = \frac{3}{2}$ and $\cot B = \frac{4}{3}$, determine the value of:

$\sin A \cos B + \csc B \cot A + \csc\left(\frac{\pi}{2} - A\right)\sin\left(\frac{\pi}{2} - B\right)$

Draw two triangles, one with angle A and one with angle B. Label the appropriate sides and find the missing length.





$= \sin A \cos B + \csc B \cot A + \csc\left(\frac{\pi}{2} - A\right)\sin\left(\frac{\pi}{2} - B\right)$

$= \frac{2}{3}\cdot\frac{4}{5} + \frac{5}{3}\cdot\frac{\sqrt{5}}{2} + \frac{3}{\sqrt{5}}\cdot\frac{4}{5} \qquad = \frac{8}{15} + \frac{5\sqrt{5}}{6} + \frac{12}{5\sqrt{5}}$

$= \frac{16\sqrt{5}}{30\sqrt{5}} + \frac{125}{30\sqrt{5}} + \frac{72}{30\sqrt{5}} = \frac{16\sqrt{5}+197}{30\sqrt{5}}$

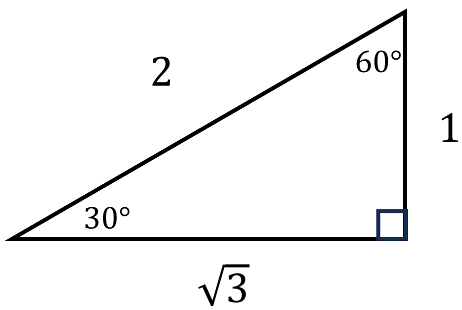
6.2 Homework

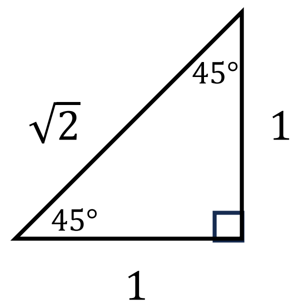
1-4 bcf…, 5ace, 6bcf, 7bd, 8bd, 9, 11a, 12, 15

6.3 Trigonometric Function – General & Special Angles

Special Triangles - 30°, 60°, 90° and 45°, 45°, 90°

These triangles are very useful because the angles and sides are exact values. They are typically involved with exact value calculations.





Trigonometric Ratios of Special Triangles

The trigonometric ratios derived from the special triangles.

$\sin 30° = \frac{1}{2}$ $\quad\sin 60° = \frac{\sqrt{3}}{2}$ $\quad\sin 45° = \frac{1}{\sqrt{2}}$

$\cos 30° = \frac{\sqrt{3}}{2}$ $\quad\cos 60° = \frac{1}{2}$ $\quad\cos 45° = \frac{1}{\sqrt{2}}$

$\tan 30° = \frac{1}{\sqrt{3}}$ $\quad\tan 60° = \sqrt{3}$ $\quad\tan 45° = 1$

The ratios are still the same if the angles were replaced with the radian equivalent angles.

$\sin\frac{\pi}{6} = \frac{1}{2}$ $\quad\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$ $\quad\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$ $\quad\cos\frac{\pi}{3} = \frac{1}{2}$ $\quad\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$\tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$ $\quad\tan\frac{\pi}{3} = \sqrt{3}$ $\quad\tan\frac{\pi}{4} = 1$

Trigonometric Ratios Using the Sine and Cosine Curve

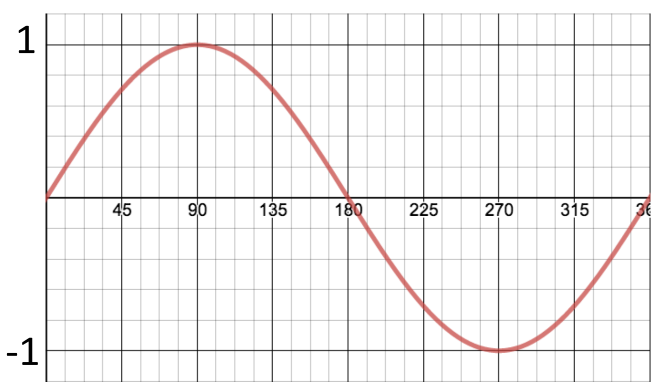
For angles 0° , 90° , 180° , 270°, and 360°, since these are all multiples of 90°, we refer to these angles as $90°n, n \in \mathbb{Z}$.

Similarly, angles 0, $\frac{\pi}{2}$, π, $\frac{3\pi}{2}$, and 2π will be referred to as $\frac{\pi}{2}n, n \in \mathbb{Z}$.

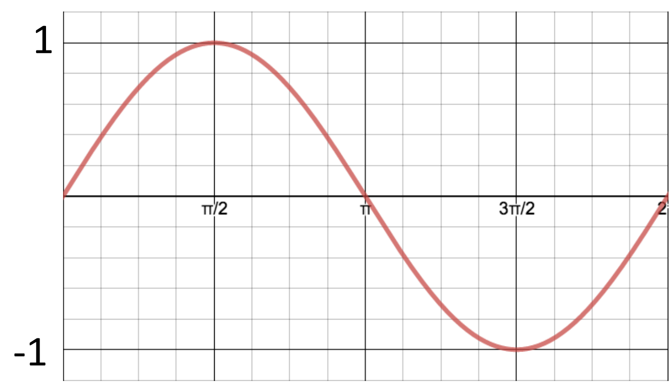
To find trigonometric ratios involving $90°n$, it is necessary to know the shape of sine and cosine functions (also true for $\frac{\pi}{2}n$).

Graph of $y = \sin x$ from $0 \leq x < 360°$ and $0 \leq x < 2\pi$

Sine curve in degrees



Sine curve in radians

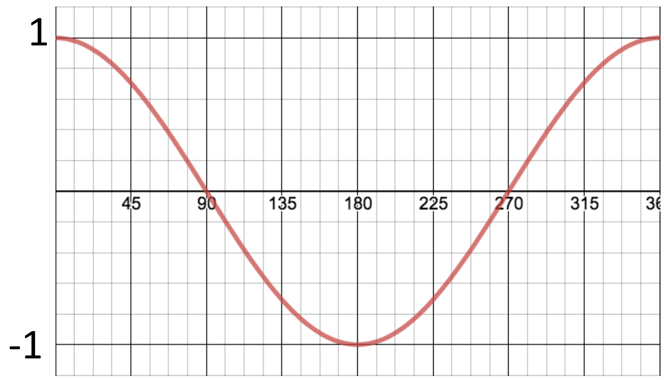


From these sine curves, we get the following trig ratios:

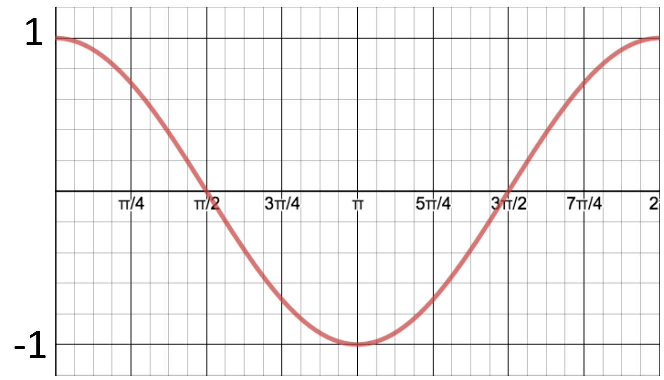
$\sin 0° = 0$	$\sin 0 = 0$
$\sin 90° = 1$	$\sin \frac{\pi}{2} = 1$
$\sin 180° = 0$	$\sin \pi = 0$
$\sin 270° = -1$	$\sin \frac{3\pi}{2} = -1$
$\sin 360° = 0$	$\sin 2\pi = 0$

Graph of $y = \cos x$ from $0 \le x < 360°$ and $0 \le x < 2\pi$

Cosine curve in degrees



Cosine curve in radians



$\cos 0° = 1$

$\cos 90° = 0$

$\cos 180° = -1$

$\cos 270° = 0$

$\cos 360° = 1$

$\cos 0 = 1$

$\cos \frac{\pi}{2} = 0$

$\cos \pi = -1$

$\cos \frac{3\pi}{2} = 0$

$\cos 2\pi = 1$

Trigonometric Ratios for Tangent

The tangent curve is harder to remember, so we use the identity: $\tan x = \frac{\sin x}{\cos x}$

$\tan 0° = \frac{\sin 0°}{\cos 0°} = \frac{0}{1} = 0$

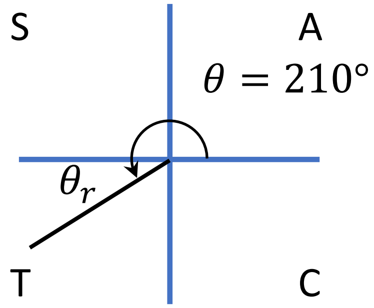
$\tan 90° = \frac{\sin 90°}{\cos 90°} = \frac{1}{0} = undefined$

$\tan 180° = \frac{\sin 180°}{\cos 180°} = \frac{0}{-1} = 0$

$\tan 270° = \frac{\sin 270°}{\cos 270°} = \frac{-1}{0} = undefined$

$\tan 360° = \frac{\sin 360°}{\cos 360°} = \frac{0}{1} = 0$

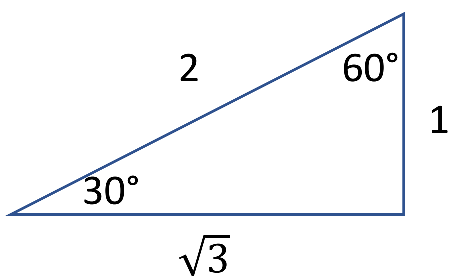
Ex. Determine exact value of $\cos 210°$ without the calculator.



Find the reference angle, and find cosine ratio

$\theta_r = 210° - 180° = 30°$ $\qquad$ $\cos 30° = \frac{\sqrt{3}}{2}$

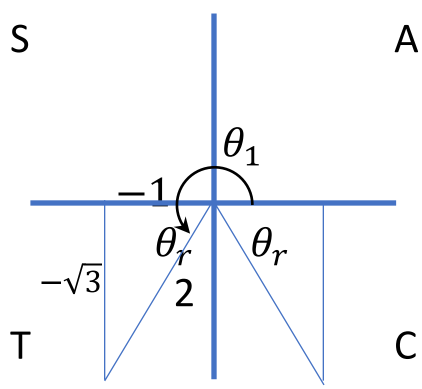
Draw the special triangle



$\therefore \cos 210° = -\frac{\sqrt{3}}{2}$

the ratio is negative because it's in QIII

Ex. Find all θ, $0° \leq \theta < 360°$ for which $\sin \theta = -\frac{\sqrt{3}}{2}$.



$\theta_r = 60°$

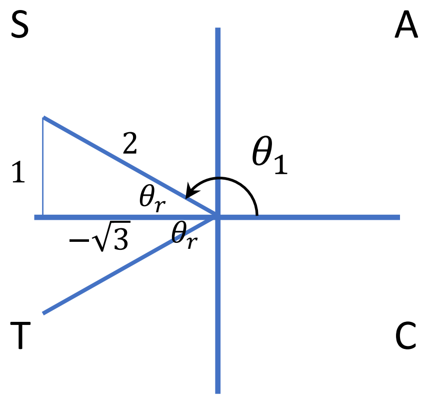
$\theta_1 = 180 + 60$ $\qquad$ $\theta_2 = 360 - 60$

$\theta_1 = 240°$ $\qquad$ $\theta_2 = 300°$

$\theta = 240°, 300°$

Ex. Find all θ, $0 \le \theta < 2\pi$ for which $\sec\theta = -\frac{2}{\sqrt{3}}$

Since $\sec\theta$ (or $\cos\theta$) is negative, the terminal arm is in II and III.



$$\sec\theta = -\frac{2}{\sqrt{3}}$$

$$\cos\theta = -\frac{\sqrt{3}}{2}$$

$$\theta_r = 30° = \frac{\pi}{6}$$

$$\theta_1 = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$$

$$\theta_2 = \pi + \frac{\pi}{6} = \frac{7\pi}{6}$$

$$\theta = \frac{5\pi}{6}, \frac{7\pi}{6}$$

6.3 Homework

1-13 bcf…, 15

6.4 Graphing Basic Trigonometric Functions

Sine Curve

$$y = a\sin b(x-c)+d$$

Period = $\frac{2\pi}{b}$

Cosine Curve

$$y = a\cos b(x-c)+d$$

Period = $\frac{2\pi}{b}$

Tangent Curve

$$y = a\tan b(x-c)+d$$

Period = $\frac{\pi}{b}$

a - vertical exp / comp / reflection over x-axis
Amplitude = $|a|$ - distance from the middle of the curve to the top (or bottom)

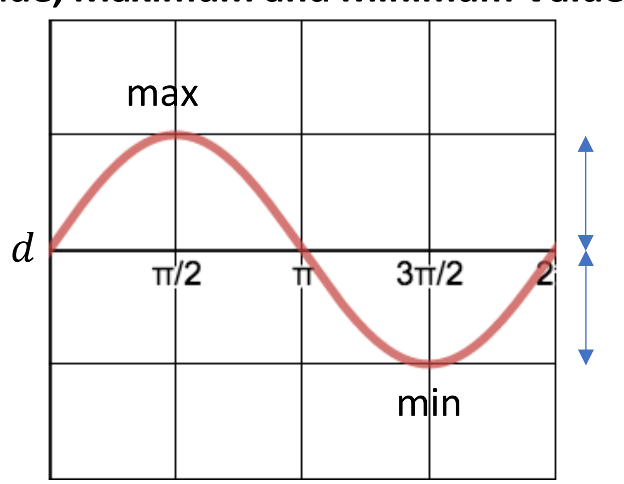
b - horizontal exp / comp / reflection over y-axis
The period of the sinusoidal function (sine or cosine) is given by the following:

$$period = \frac{2\pi}{|b|} \quad \text{which means} \quad b = \frac{2\pi}{period}$$

c - horizontal translation (phase shift)

d - vertical translation (vertical displacement)

Amplitude, Maximum and Minimum Values

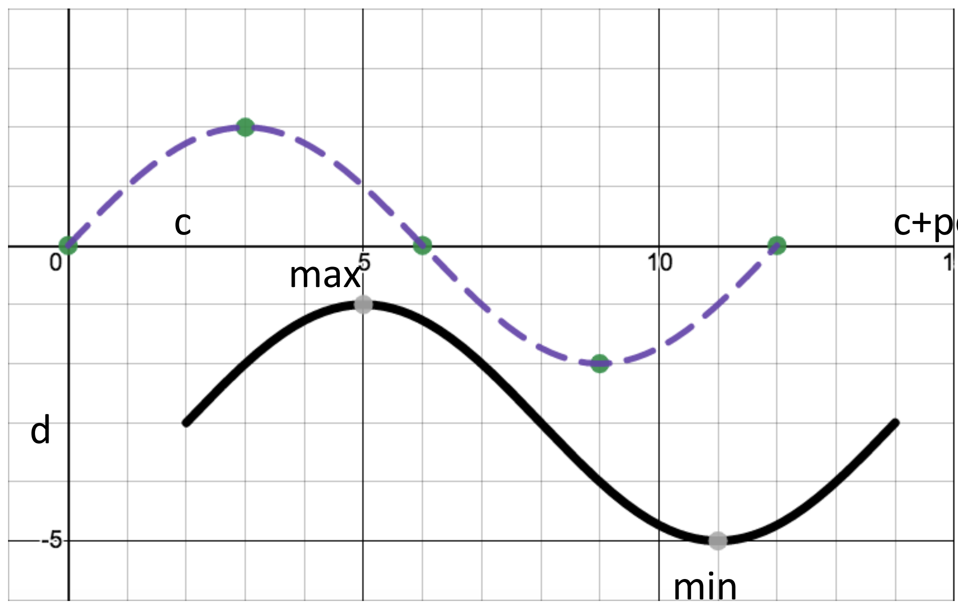


$$amp = \left|\frac{max-min}{2}\right| \qquad d = \frac{max+min}{2}$$

Ex. Graph $y = 2\sin\frac{\pi}{6}(x-2) - 3$

$amp = 2$ $\quad$ $period = \frac{2\pi}{\frac{\pi}{6}} = 12$ $\quad$ $ps = 2$ $\quad$ $vd = -3$

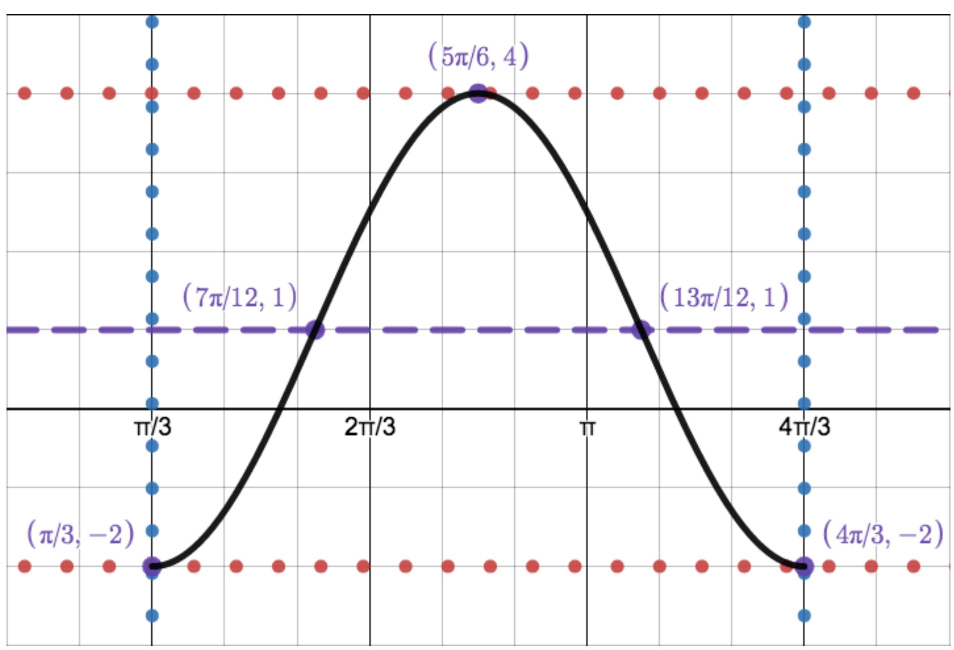
$max = -1$ $\quad$ $min = -5$



Ex. Graph $y = -3\cos 2\left(x - \frac{\pi}{3}\right) + 1$

$amp = |-3| = 3$ $\quad$ $p = \frac{2\pi}{2} = \pi$ $\quad$ $ps = \frac{\pi}{3}$

$vd = 1$ $\quad$ $max = 4$ $\quad$ $min = -2$



#3a pg 284

$y = \frac{1}{3}\sin\left(2x + \frac{\pi}{3}\right) - 1 \quad \rightarrow \quad y = \frac{1}{3}\sin\left(2\left(x + \frac{\pi}{6}\right)\right) - 1$

amp = $\frac{1}{3}$

phase shift = $-\frac{\pi}{6}$

period = $\frac{2\pi}{2} = \pi$

vertical displacement = -1

max = $-1 + \frac{1}{3} = -\frac{2}{3}$

begin point $\left(-\frac{\pi}{6}, -1\right)$

min = $-1 - \frac{1}{3} = -\frac{4}{3}$

end point $\left(-\frac{\pi}{6} + \pi, -1\right) = \left(\frac{5\pi}{6}, -1\right)$

middle point $\left(\frac{2\pi}{6}, -1\right)$

#5a

$y = a \sin b\,(x - c)$ and $y = a \cos b\,(x - c)$

Sine:

$amp = \frac{max - min}{2} = \frac{3-(-3)}{2} = 3 \qquad \therefore a = 3$

Begin $x = 0$ Ends $x = 4$ $\rightarrow$ $period = 4 - 0 = 4$

$b = \frac{2\pi}{4} = \frac{\pi}{2}$

$y = 3\sin\frac{\pi}{2}x$

Cosine:

$y = 3\cos\frac{\pi}{2}(x - c)$

Begin $x = 1$ $\therefore c = 1$

$y = 3\cos\frac{\pi}{2}(x - 1)$

Ex. Graph $y = -3\sin\frac{\pi}{3}(x+2)+1$

$a = -3$ $b = \frac{\pi}{3}$ $c = -2$ $d = 1$

$(x,y) \rightarrow \left(\frac{1}{b}x + c, ay + d\right)$
$(x,y) \rightarrow \left(\frac{3}{\pi}x - 2, -3y + 1\right)$

$(0,0) \rightarrow \left(\frac{3}{\pi}(0) - 2, -3(0) + 1\right) = (-2, 1)$

$\left(\frac{\pi}{2}, 1\right) \rightarrow \left(\frac{3}{\pi}\left(\frac{\pi}{2}\right) - 2, -3(1) + 1\right) = \left(-\frac{1}{2}, -2\right)$

$(\pi, 0) \rightarrow \left(\frac{3}{\pi}(\pi) - 2, -3(0) + 1\right) = (1, 1)$

$\left(\frac{3\pi}{2}, -1\right) \rightarrow \left(\frac{3}{\pi}\left(\frac{3\pi}{2}\right) - 2, -3(-1) + 1\right) = \left(\frac{5}{2}, 4\right)$

$(2\pi, 0) \rightarrow \left(\frac{3}{\pi}(2\pi) - 2, -3(0) + 1\right) = (4, 1)$

amp = 3 period= $\frac{2\pi}{\frac{\pi}{3}} = 6$ ps = -2 vd = 1

max = 1 + 3 = 4 begin at x = -2

min = 1- 3 = -2 ends at x = -2 + 6 = 4

6.4 Homework
1, 2, 3bcf, 4, 5 bcf…, 6, 9, 10

6.5 Applications of Periodic Functions

Ex. A weight is attached to a spring and set in motion by stretching the spring and releasing it. The distance (cm) the spring is from its rest position at time t (sec) is given by the equation $d = 5\sin(4\pi\, t)$

a) How many cycles per second does the spring make?

$b = 4\pi$ $\qquad\qquad period = \frac{2\pi}{b} = \frac{2\pi}{4\pi} = \frac{1}{2}$

If period is 0.5 sec, then in one second there are 2 cycles.

b) Graph the motion of the spring for one period.

$d = 5sin(4\pi\, t)$

The amplitude is 5.

Since the vertical displacement is 0, the max =0+5=5, and min = 0-5=-5

c) At what time will the first max and min extremes of the cycles occur?

Max occurs at ¼ of the period, while min occurs ¾ of the period.

Max at $t = 0.125$ sec $\qquad\qquad$ Min at $t = 0.375$ sec

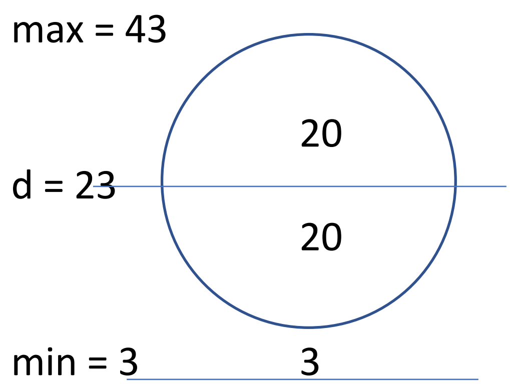
Ex. The voltage E of an electrical circuit has an amplitude of 220 volts and a frequency of 60 cycles per second. If E = 220 when t = 0, find a periodic equation in terms of cosine that describes this voltage.

60 cycles per second → 1 cycle takes $\frac{1}{60}$ second

Period is $\frac{1}{60}$ sec, $\qquad \therefore b = \frac{2\pi}{\frac{1}{60}} = 120\pi$

$E = 220\cos 120\pi(t)$

Ex. A Ferris wheel has a radius of 20 m and rotates every 60 seconds. A rider enters the seat at the lowest point of the Ferris wheel, 3 m above the ground. Find a cosine function that gives the height h, after t seconds of motion for the rider and find at what time the rider first reaches a height of 30 m.



radius of 20 m → amp = 20

Rotates every 60 sec → period = 60

Period = 60 → $b = \frac{2\pi}{60} = \frac{\pi}{30}$

Lowest point is 3 m → min = 3

Min = 3 and amp = 20 → vertical displacement = 20 + 3 = 23

a. Write the equation:

$$h = -20\cos\frac{\pi}{30}t + 23$$

b. Solve for t when $h = 30$

$$30 = -20\cos\frac{\pi}{30}t + 23$$

$$7 = -20\cos\frac{\pi}{30}t$$

$$-\frac{7}{20} = \cos\frac{\pi}{30}t$$

Since cosine ratio is negative, solution is in QII or QIII

$$\frac{\pi}{30}t = \cos^{-1}\left(-\frac{7}{20}\right)$$

$$\frac{\pi}{30}t = 1.928$$

this solution is in QII (while 4.355 is in QIII; first occurrence is in QII)

$$t = 18.4$$

The rider first reaches 30 m at 18.4 seconds.

Ex. The following equation describes the temperature of a city in Celsius: $T = 35\sin\left[\left(\frac{2\pi}{365}\right)(x-100)\right] + 27$. When $x = 1$, the day is Jan. 1 and $x = 365$ it is December 31. Find the days the temperature is 0°C.

Solve for when the temperature equals 0.

$$0 = 35\sin\left[\left(\frac{2\pi}{365}\right)(x-100)\right] + 27$$

Solve for x

$$-27 = 35\sin\left[\frac{2\pi}{365}(x-100)\right]$$

$$-\frac{27}{35} = \sin\left[\frac{2\pi}{365}(x-100)\right]$$

Let $\theta = \frac{2\pi}{365}(x-100)$

$$\sin\theta = -\frac{27}{35}$$

sine ratio is negative; the solutions are in QIII and QIV

$$\theta = \sin^{-1}\left(-\frac{27}{35}\right)$$

$$\theta = -0.881083173$$

solution is in QIV, find positive co-terminal answer

θ in quadrant IV:

$$\theta_2 = -0.88103173 + 2\pi = 5.40210213418$$

Find reference angle, θ_r

$$\theta_r = 2\pi - 5.40210213418 = 0.88103173$$

θ in quadrant III:

$$\theta_1 = \pi + 0.88103173$$

$$\theta_1 = 4.02267582659$$

$$\theta = 4.02267582659, 5.40210213418$$

For both solutions for θ, use $\theta = \frac{2\pi}{365}(x - 100)$ and solve for x

$\frac{2\pi}{365}(x - 100) = 4.023$

$x - 100 = 4.023 \times \frac{365}{2\pi}$

$x = 234 + 100 = 334$

$\frac{2\pi}{365}(x - 100) = 5.402$

$x - 100 = 5.402 \times \frac{365}{2\pi}$

$x = 414$

414 – 365 = 49

The city is below 0° on day 334 until day 49 of next year

Day 334 → Nov. 30

Day 49 → Feb. 18

The city is at 0°C or lower between Nov. 30 and Feb. 18

Ex. A Ferris wheel has a radius of 25 m and rotates every 80 seconds. A rider enters the seat at the lowest point of the Ferris wheel 2 metres above the ground.

a. Write a sinusoidal function that models the position of the Ferris wheel seat, that begins at the bottom. h for height in metres, and t for time in seconds

amp = 25 seat starts at bottom, $a = -25$

period = 80 $b = \frac{2\pi}{80} = \frac{\pi}{40}$

no phase shift

min = 2, max = 52 $d = \frac{52+2}{2} = 27$

$h = -25\cos\frac{\pi}{40}t + 27$

b. Determine the height of the seat at $t = 26$ seconds

$$h = -25\cos\left[\frac{\pi}{40}(26)\right] + 27$$

$$h = 38.3 \text{ m}$$

c. Determine at what times will the seat be at 42 m in the first cycle.

$$42 = -25\cos\frac{\pi}{40}t + 27$$

$$15 = -25\cos\frac{\pi}{40}t$$

$$\cos\frac{\pi}{40}t = -0.6$$

Let $\theta = \frac{\pi}{40}t$

$\frac{\pi}{40}t = \cos^{-1}(-0.6) = 2.214$ — cosine ratio is negative, the solutions in QII and QIII

$\theta_1 = 2.214$ — This solution is in QII

Find the reference angle θ_r

$$\theta_r = \pi - 2.214 = 0.927$$

Find the solution in QIII

$$\theta_2 = \pi + 0.927$$

$$\theta_2 = 4.069$$

Solving for t for both solutions of θ

$\frac{\pi}{40}t = 2.214$	$\frac{\pi}{40}t = 4.069$
$t = 28.2$ s	$t = 51.8$ s

The seat would be at a height of 42 m at 28.2 s and 51.8 s.

d. Determine how long the seat is above 42 m.

Take the difference between the two times from part c

$$t = 51.8 - 28.2$$

$$t = 23.6 \text{ s}$$

The seat spends 23.6 s above 42 m.

6.5 Homework

5, 6, 9, 10