1.3 Geometric Sequences

In an arithmetic sequence, a constant number is added to each term to get the next term. In a geometric sequence, each term is multiplied by a constant number to get the next term.

Consider the sequence $3, 6, 12, 24, \ldots$ Multiplying each term by 2 results in the next term. This number is called the **common ratio**, because the ratio value is found by dividing any term by the preceding term.

In the sequence $3, 6, 12, 24, \ldots,\ 6 \div 3 = 2,\ 12 \div 6 = 2,\ 24 \div 12 = 2, \ldots$

Geometric Sequence

A sequence is geometric if the ratio of consecutive terms is constant.

$a_1, a_2, a_3, \ldots, a_n$ is geometric if there is a number $r,\ r \neq 0$ such that $\frac{a_2}{a_1} = r,\ \frac{a_3}{a_2} = r, \ldots, \frac{a_n}{a_{n-1}} = r$

The number r is called the **common ratio** of the geometric sequence.

Example 1 For each geometric sequence, find the common ratio.

a) $2,\ 6,\ 18,\ 54, \ldots$

b) $3,\ -6,\ 12,\ -24, \ldots$

c) $-8,\ -4,\ -2,\ -1, \ldots$

►*Solution*: **a)** $\frac{6}{2} = \frac{18}{6} = \frac{54}{18} = 3 \ \rightarrow \ r = 3$

b) $\frac{-6}{3} = \frac{12}{-6} = \frac{-24}{12} = -2 \ \rightarrow \ r = -2$

c) $\frac{-4}{-8} = \frac{-2}{-4} = \frac{-1}{-2} = \frac{1}{2} \ \rightarrow \ r = \frac{1}{2}$

Deriving the Formula for the *n*th Term of a Geometric Sequence

let $a =$ the first term, $r =$ the common ratio

a_1
$a_2 = a_1 \cdot r$
$a_3 = a_2 \cdot r = (a_1 \cdot r) \cdot r = a_1 \cdot r^2$
$a_4 = a_3 \cdot r = (a_1 \cdot r^2) = a_1 \cdot r^3$
$\vdots$
$a_n = a_1 \cdot r^{n-1}$

Note: The exponent in each term is one less than the subscript of the term. In general $a_m = a_n r^{(m-n)}$.

The *n*th Term of a Geometric Sequence

The *n*th term of a geometric sequence with common ratio r has the form $t_n = ar^{n-1}$, for any integer $n \geq 1$

Example 2 Find the 8th term of the geometric sequence $3,\ 12,\ 48,\ 192, \ldots$

►*Solution*: The common ratio is $\frac{12}{3} = 4$, with $a = 3$

$$t_n = ar^{n-1}$$
$$t_8 = 3(4)^{8-1}$$
$$= 3(4)^7$$
$$= 49152$$

Example 3 The 4th term of a geometric sequence is 125, and the 9th term is $\frac{125}{32}$. Find the 13th term.

►*Solution*: The 4th term $t_4 = ar^3 = 125$. The 9th term $t_9 = ar^8 = \frac{125}{32}$

$$ar^8 = (ar^3) \cdot r^5 = \frac{125}{32}$$
$$125 \cdot r^5 = \frac{125}{32}$$
$$r^5 = \frac{1}{32}$$
$$r = \frac{1}{2}$$

$$ar^3 = 125$$
$$a\left(\frac{1}{2}\right)^3 = 125$$
$$a = 1000$$

The 13th term is $t_{13} = 1000\left(\frac{1}{2}\right)^{12} = \frac{125}{512}$

Example 4 What value of x in $x,\ 2x + 2,\ 3x + 3$ will form a geometric sequence?

►*Solution*: $r = \frac{2x+2}{x} = \frac{3x+3}{2x+2}$

$$4x^2 + 8x + 4 = 3x^2 + 3x$$
$$x^2 + 5x + 4 = 0$$
$$(x + 4)(x + 1) = 0$$
$$x = -1,\ -4$$

1.3 Exercise Set

1. Determine if the sequence is geometric. If it is, find the common ratio.

a) $4,\ 12,\ 36,\ 72,\ldots$ _____

b) $3,\ 12,\ 48,\ 142,\ldots$ _____

c) $1,\ -\frac{1}{2},\ \frac{1}{4},\ -\frac{1}{8},\ldots$ _____

d) $1,\ -1,\ 1,\ -1,\ldots$ _____

e) $3,\ -6,\ -12,\ 24,\ldots$ _____

f) $1,\ \frac{1}{2},\ \frac{1}{3},\ \frac{1}{4},\ldots$ _____

g) $\frac{1}{4},\ \frac{1}{6},\ \frac{1}{9},\ \frac{2}{27},\ldots$ _____

h) $\frac{2}{5},\ -\frac{2}{3},\ \frac{10}{9},\ -\frac{50}{27},\ldots$ _____

i) $3x^2,\ 12x^4y^3,\ 48x^6y^6,\ldots$ _____

j) $\sqrt{2},\ \sqrt{6},\ 3\sqrt{2},\ 3\sqrt{6},\ldots$ _____

2. Write the first five terms of the geometric sequence.

a) $1,\ 4,\ \_\_\_\_,\ \_\_\_\_,\ \_\_\_\_$

b) $1,\ \_\_\_\_,\ 4,\ \_\_\_\_,\ \_\_\_\_$

c) $\frac{1}{2},\ \_\_\_\_,\ \_\_\_\_,\ \frac{1}{16},\ \_\_\_\_$

d) $4,\ \_\_\_\_,\ \_\_\_\_,\ -13.5,\ \_\_\_\_$

e) $\_\_\_\_,\ 54,\ 18,\ \_\_\_\_,\ \_\_\_\_$

f) $1,\ \_\_\_\_,\ 3,\ \_\_\_\_,\ \_\_\_\_$

g) $3,\ \_\_\_\_,\ 3^{2x+1},\ \_\_\_\_,\ \_\_\_\_$

h) $1,\ \_\_\_\_,\ x^4,\ \_\_\_\_,\ \_\_\_\_$

i) $5,\ 5^{2x-1},\ \_\_\_\_,\ \_\_\_\_,\ \_\_\_\_$

j) $1,\ -\frac{x}{3},\ \_\_\_\_,\ \_\_\_\_,\ \_\_\_\_$

3. Find all possible values of r for a geometric sequence with the two given terms.

a) $a_5 = 5,\ a_7 = 25$

b) $a_2 = 4,\ a_6 = \frac{1}{4}$

c) $a_4 = 2\sqrt{2},\ a_7 = 8$

d) $a_3 = 1,\ a_6 = \sqrt{2}$

4. Find the indicated value using the information given.

a) a_{11}, if $a_1 = \frac{1}{128},\ r = 2$

b) a_9, if $a_1 = 3,\ a_2 = \sqrt{3}$

c) a_{42}, if $a_{40} = 9,\ a_{41} = 36$

d) a_9, if $a_4 = 5,\ a_6 = 20$

e) n, if $a_1 = 729,\ a_2 = 243,\ l = \frac{1}{9}$

f) n, if $a_1 = 2048,\ a_2 = 1024,\ l = 1$

g) a_1, if $a_5 = 27,\ r = 3$

h) a_1, if $a_7 = 128,\ r = 4$

i) r, if $a_{10} = 25,\ a_{12} = 225$

j) r, if $a_{25} = 12,\ a_{31} = 96$

k) a_8, if $a_n = 3a_{n-1},\ a_1 = \frac{1}{27}$

l) a_6, if $a_n = 0.1a_{n-1},\ a_1 = 1000$

5. Insert two geometric means between a and b.

6. Given the geometric sequence $a,\ \frac{a}{b},\ \frac{a}{b_2},\ldots$ determine an expression for $t_n - t_{n-1},\ n > 2$.

7. Find x so that $x-1,\ x$, and $x+2$ are consecutive terms of a geometric sequence.

8. Find the common ratio r for the geometric sequence $x-2,\ 5-x,\ 5x-7,\ldots$

9. What number must be added to $-2,\ 4,\ 19$ so that the resulting numbers are three terms of a geometric sequence?

10. If the first two terms of a geometric sequence are $\sqrt{2}$, and $\sqrt[3]{2}$, what is the fourth term?

11. If the product of the first three terms of a geometric is -8, and the sum is $\frac{14}{3}$, what is the common ratio of the sequence?

12. In the sequence $3,\ x,\ y,\ 25$, the first three terms form an arithmetic sequence, and the last three terms form a geometric sequence. Find x and y.

13. The enrolment at Earl Marriott Secondary in Surrey, BC, was 400 in 1973. If the school's population has increased by 5% a year, how many students will be going to the school in 2010?

14. If a starting salary is $28 000, and one expects to receive an annual increase of 6%, what is the salary at the beginning of the eighth year of work?

15. With each cycle, a vacuum pump removes 25% of the air in a glass container. What percent of the air has been removed after 10 cycles?

16. A car costs a company $40 000. Each year, the car depreciates 16% of its value. What is the value of the car after five years?

17. Initially, a pendulum swings through an arc of 45 cm. On each successive swing, the length of the arc decreases by 2% of the previous length. What is the length of the arc after 12 swings?

18. A ball is dropped from a height of 10 metres. Each time it strikes the ground it bounces up 75% of its previous height. How many bounces does the ball need before the bounce is less than 20 cm high?

19. Here is a diagram of the steps taken to get strawberries from the farmer to the consumer: *Farmer → Trucker → Regional Market → Trucker → Wholesaler → Trucker → Retailer → Consumer*. If the farmer gets 75 cents per kilogram, and if each person in the chain makes a 20% profit, how much does the consumer pay?

20. A truck radiator contains fifty litres of water. Five litres of water is removed and replaced with pure antifreeze; then five litres of the mixture is removed and replaced. How much antifreeze is in the radiator after this process is repeated five times?